Three-dimensional waves in a two-fluid system generated by a moving pressure

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1. Introduction

We consider a three-dimensional steady wave problem in which a distribution of pressure moves at constant velocity U on the interface between two fluids, with a lighter fluid lying above a heavier one. The lower fluid is of infinite depth and the upper fluid layer is bounded above by a free surface. This models, in an inverse way, interfacial flows past submerged bodies on the interface. The pressure can also be applied on the free surface, instead of the interface. The three dimensional problem is formulated as a nonlinear integro-differential equation by using the Green's identity in each layer and the dynamic boundary conditions.

In the last half century there were a number of research papers considering forced waves which propagate at the interface of two-fluids (see for example Hudimac [4], Crapper [2], Keller and Munk [5], Yih and Zhu [10], Tulin and Miloh [7], Avital and Miloh [1], Wei et al [8], Yeung and Nguyen [9]). Most of them are studying the linear problem. It was shown that the wave system at the interface or at the free surface is determined by a surface-wave mode and an internal-wave mode.

2. Formulation and numerical scheme

The fluids are assumed to be inviscid and incompressible and the flow to be irrotational in both layers. There is no shear between the layers and the free-surface and interfacial tensions are neglected. The subscript 1 refers to the lower fluid, and the subscript 2 refers to the upper layer. The lower layer is of infinite depth and the upper fluid, which is bounded above by a free surface, has a rest equilibrium thickness h. We choose a frame of reference moving with the disturbance. We introduce Cartesian coordinates x, y, z with the z-axis directed vertically upwards with $z = 0$ in the still interface and the x-axis in the opposite direction of the velocity U and we denote by $z = \zeta(x, y)$ the position of the interface and by $z = \zeta_s(x, y)+h$ the equation of the free-surface.The governing equations in each layer are

$$\Delta\Phi_i = 0\,, \quad i = 1, 2\,, \tag{1}$$

where Φ_i is the velocity potential in layer i. On the interface $z = \zeta(x, y)$ the full nonlinear kinematic and dynamic conditions are applied

$$\Phi_{ix}\zeta_x + \Phi_{iy}\zeta_y = \Phi_{iz}, \quad i = 1, 2, \tag{2}$$

$$\frac{1}{2}\rho_1(\Phi_{1x}^2 + \Phi_{1y}^2 + \Phi_{1z}^2) - \frac{1}{2}\rho_2(\Phi_{2x}^2 + \Phi_{2y}^2 + \Phi_{2z}^2) + (\rho_1 - \rho_2)g\zeta + p = \frac{1}{2}(\rho_1 - \rho_2)U^2. \tag{3}$$

Here ρ_i are the densities, g is the acceleration due to gravity, and p is the pressure distribution which models the disturbance at the interface.

On the free-surface $z = \zeta_s(x, y) + h$ we have again the kinematic and dynamic conditions

$$\Phi_{2x}\zeta_x + \Phi_{2y}\zeta_y = \Phi_{2z}, \tag{4}$$

$$\frac{1}{2}(\Phi_{2x}^2 + \Phi_{2y}^2 + \Phi_{2z}^2) + g(\zeta_s + h) = \frac{1}{2}U^2 + gh. \tag{5}$$

The velocity and radiation boundary conditions at infinity are

$$(\Phi_{1x}, \Phi_{1y}, \Phi_{1z}) \to (U, 0, 0) \quad \text{for } z \to -\infty \tag{6}$$

$$\text{no waves} \quad \text{for } x \to -\infty. \tag{7}$$

We present results by choosing the pressure as

$$p(x,y) = \begin{cases} P_0 e^{\frac{L^2}{x^2-L^2}+\frac{L^2}{y^2-L^2}}, & |x| < L \text{ and } |y| < L \\ 0 & \text{otherwise.} \end{cases}$$

Here P_0 is a constant and L defines the size of the support of the pressure. We introduce dimensionless variables by using U as the unit velocity and L as the unit length.

Combining equations (2) and (3), respectively (4) and (5), and using the chain rule we obtain

$$\frac{1}{2}\frac{(1+\zeta_x^2)\phi_{1y}^2 + (1+\zeta_y^2)\phi_{1x}^2 - 2\zeta_x\zeta_y\phi_{1x}\phi_{1y}}{1+\zeta_x^2+\zeta_y^2}$$

$$-\frac{R}{2}\frac{(1+\zeta_x^2)\phi_{2y}^2 + (1+\zeta_y^2)\phi_{2x}^2 - 2\zeta_x\zeta_y\phi_{2x}\phi_{2y}}{1+\zeta_x^2+\zeta_y^2} + \frac{1-R}{F^2}\zeta + \varepsilon P = \frac{1-R}{2}, \tag{8}$$

$$\frac{1}{2}\frac{(1+\zeta_{s_x}^2)\phi_{s_y}^2 + (1+\zeta_{s_y}^2)\phi_{s_x}^2 - 2\zeta_{s_x}\zeta_{s_y}\phi_{s_x}\phi_{s_y}}{1+\zeta_{s_x}^2+\zeta_{s_y}^2} + \frac{\zeta_s}{F^2} = \frac{1}{2} \tag{9}$$

where $\phi_i(x,y) = \Phi_i(x,y,\zeta(x,y))$, $\phi_s = \Phi_2(x,y,\zeta_s(x,y)+H)$, $F = U/(gL)^{1/2}$ is the Froude number, $R = \rho_2/\rho_1$ is the density ratio, $H = h/L$ is the relative thickness of the upper layer and $\varepsilon = \dfrac{P_0}{\rho_1 U^2}$ is the dimensionless magnitude of the pressure.

The numerical scheme is an extension to a two-fluid system of that used by Părău and Vanden-Broeck [6] for the computation of forced gravity waves in deep water. It is based on a boundary integral equation method introduced by Forbes [3] for three-dimensional gravity free-surface flows past a source.

The formulation involves applying in each fluid Green's second identity for the functions $\Phi_i - x$ $(i = 1,2)$ and G, where $G(P,P^*)$ is the three dimensional free space Green function

$$G(P,P^*) = \frac{1}{4\pi}\frac{1}{([x-x^*]^2+[y-y^*]^2+[z-z^*]^2)^{1/2}}, \tag{10}$$

for the volumes V_1 and V_2 and $P = (x,y,z)$, $P^* = (x^*,y^*,z^*) \in S_I$. The volume V_1 consists of a large-radius hemisphere bounded above by the interface S_I, except for a small hemisphere around the point P^* and the volume V_2 consists of a large-radius cylinder bounded by the free surface S_F and the interface S_I, except a small hemisphere around the point P^*. Another integral equation is obtained when P^* is on the free-surface S_F and we apply the Green's second identity on the volume V_2', which is similar with V_2, the difference being that the small hemisphere excepted around the point is now on the free surface. We obtain the following equations

$$\frac{1}{2}(\Phi_1(P^*)-x^*) = \int_{\mathbf{S_I}} (\Phi_1(P)-x)\frac{\partial G(P,P^*)}{\partial n_1} - G(P,P^*)\frac{\partial(\Phi_1(P)-x)}{\partial n_1} dS \quad \text{for } P^* \in S_I,$$

$$\frac{1}{2}(\Phi_2(P^*)-x^*) = \int_{\mathbf{S_I}} (\Phi_2(P)-x)\frac{\partial G(P,P^*)}{\partial n_2} - G(P,P^*)\frac{\partial(\Phi_2(P)-x)}{\partial n_2} dS_P +$$

$$\int_{\mathbf{S_F}} (\Phi_2(P)-x)\frac{\partial G(P,P^*)}{\partial n_s} - G(P,P^*)\frac{\partial(\Phi_2(P)-x)}{\partial n_s} dS_P, \quad \text{for } P^* \in S_I \text{ and } P^* \in S_F,$$

where n_i $(i = 1, 2)$ is the normal vector at the interface pointing into fluid i and n_s is the normal vector at the free surface.

The integro-differential equations are projected onto the Oxy plane and the singularities on the integrals may be isolated by addition and subtraction of a quantity, which can be evaluated in closed form (see Forbes [3] for details).

For the numerical scheme we truncate the intervals $-\infty < x < \infty$ and $0 < y < \infty$ to $x_1 < x < x_N$, and $y_1 < y < y_M$ and introduce the mesh points $x_k = (k-1)\Delta x$, $k = 1, \ldots, N$ and $y_j = (j-1)\Delta y$, $j = 1, \ldots, M$. The $5NM$ unknowns are the values of $\zeta_x, \zeta_{s_x}, \phi_{1x}, \phi_{2x}, \phi_{s_x}$ at the mesh points. The integrals and the Bernoulli equation are evaluated at the points $(x_{k+1/2}, y_j)$, $k = 1, ..., N-1$, $j = 1, ..., M$ so we have $5(N-1)M$ equations. Another $5M$ equations are obtained from the radiation condition. The values of ζ and ζ_s are obtained by integrating ζ_x and ζ_{s_x} with respect to x by the trapezoidal rule. The values of the dependent variables at midpoints are found by interpolation and the derivatives are calculated by using finite differences. The $5NM$ nonlinear equations are solved by a modified Newton's method. In most computations, we used $N = 50, M = 20, \Delta x = \Delta y = 0.4$.

3. Results

The numerical scheme calculated solutions for several values of the Froude number F, density ratio R, the relative thickness of the upper layer H, and of the pressure parameter ε. The pressure was applied either at the interface, or on the free surface. The wave patterns for $F = 0.7, R = 0.9, H = 0.3, \varepsilon = 1$ in the two cases are shown in Figure 1. As noted by Yeung and Nguyen [9] on their linear study, the surface wave elevation and the interfacial wave elevation can each be composed of different wave systems: divergent waves and transverse waves due to the surface wave mode and the divergent and transverse waves due to the internal wave mode.

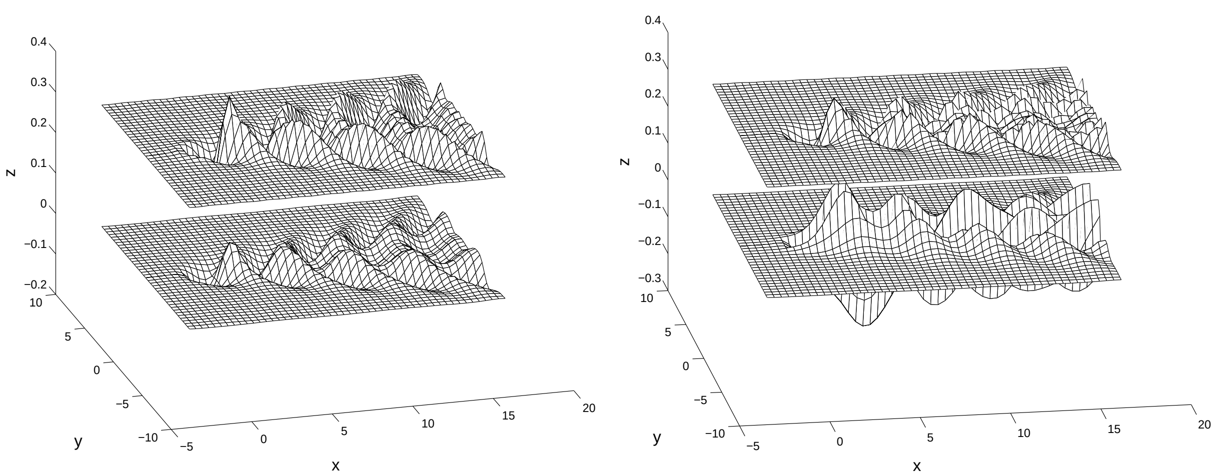

Figure 1: (a) The free surface waves and the interfacial waves generated by the applied pressure moving on the free surface inside $|x| < 1$ and $|y| < 1$. Here $F = 0.7, R = 0.9, H = 0.3, \varepsilon = 1$. (b) As (a) but with the waves generated by the applied pressure moving on the interface.

When H is increased, the amplitude of the waves on the interface (when the pressure is applied on the free surface), or on the free surface (when the pressure is applied on the interface) decreases.

The influence of the surface-wave mode on the interfacial waves can be observed when the free surface is replaced by a rigid lid. In Figure 2 we compared the contours of the interfacial waves for the two cases.

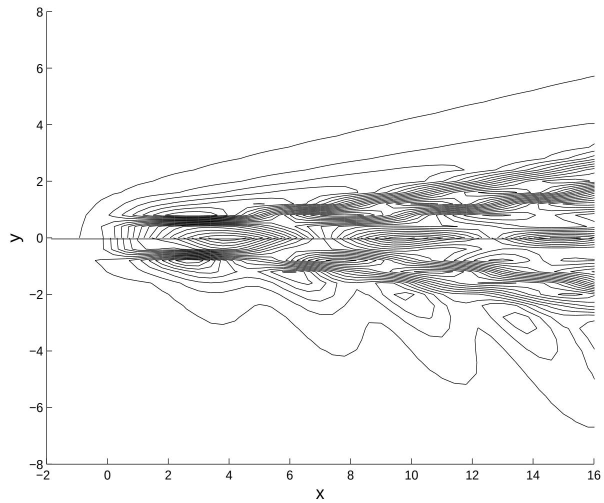

Figure 2: The interfacial waves generated by a pressure moving on the interface, when the free surface is present (lower half) and when the rigid-lid approximation is used (upper half). The contours are curves of constant vertical displacement for the same heights in both halves.

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'Three-dimensional waves in a two-fluid system generated by a moving pressure'

Discusser - T. Miloh

I am curious to know if your non-linear code applied to a two-fluid model can indeed generate solitons (Benjamin-Ono type) for an infinitely deep lower layer?

I think that the method of removing the singularity from the integral equations that you attributed to L Forbes was first suggested by L Landweber 25 years earlier.

Reply:

We thank Professor Miloh for his interest in our work.

Our numerical code can be used to compute solutions within an infinite lower layer and this would indeed provide a 3D extension of the prediction given by 2D equation of Benjamin-Ono type.

We are aware of the contribution of Landweber & Macagmo (1969), which is also a reference in Forbes (1989).