A Lagally formulation of the wave drift force

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There is no theorem like the Lagally theorem (Georg Weinblum[1])

1 Introduction

There are two well-known formulations of the wave drift force: the "far-field method", based on momentum considerations, introduced by Maruo (1960) and later extended by Newman (1967), and the "near-field method", based on direct pressure integration and first proposed by Pinkster & Van Oortmerssen (1977). The far-field method is considered as more accurate but it delivers the horizontal components only and it cannot provide individual drift forces in a multiple body configuration. The near-field method does not suffer from these restrictions but its numerical accuracy is poor in some cases.

In the software Diodore the drift force is computed following another way, based on the Lagally theorem. This method was first proposed by Guével & Grekas (1981). In their paper the (lengthy) derivations deal with an unbounded fluid domain and the drift force expressions with a free surface are given *ex abrupto*, without justifications. Moreover they are ambiguous. Subsequently, in his 1986 monograph on wave energy recovery, Guével gives the same expressions (10) (11) as we obtain here (and as have been coded in Diodore for many years), but again without proof.

In this paper we re-establish rigorously the Lagally formulation of the drift force. As compared with the classical formulations it appears to offer several advantages.

2 Theoretical developments

We consider a floating body submitted to incoming regular waves with frequency ω. The linearized velocity potential is $\Phi(x, y, z, t) = \Phi_I + \Phi_P$ where Φ_P combines diffracted and radiated components.

Be S the wetted surface of the body, Σ a surrounding fictitious control surface and F the free surface in-between. We denote by Γ_S and Γ_Σ the respective intersections of S and Σ with the free surface. The normal vector $\overrightarrow{n}$ is taken in the outward direction from the enclosed fluid domain.

The following identity holds:

$$\iint_{S\cup\Sigma\cup F} \left(\frac{1}{2}\, \nabla\Phi \cdot \nabla\Phi \; \overrightarrow{n} - \nabla\Phi \; \nabla\Phi \cdot \overrightarrow{n} \right) \, \mathrm{d}S \equiv \overrightarrow{0}. \tag{1}$$

Restricting ourselves to the horizontal components, the integral over the free surface F can be transformed as follows:

$$\iint_F - \begin{pmatrix} \Phi_x \\ \Phi_y \end{pmatrix} \Phi_z \, \mathrm{d}S = -\frac{\omega^2}{g} \iint_F \begin{pmatrix} \Phi_x \\ \Phi_y \end{pmatrix} \Phi \, \mathrm{d}S = -\frac{\omega^2}{2\,g} \int_{\Gamma_S \cup \Gamma_\Sigma} \Phi^2 \; \overrightarrow{n}_0 \, \mathrm{d}\Gamma \tag{2}$$

with $\overrightarrow{n}_0$ the unit normal vector to Γ_S and Γ_Σ within the free surface plane (thus different from $\overrightarrow{n}$ if S and/or Σ are not vertical at the free surface).

So we can write

$$-\frac{\omega^2}{2\,g} \int_{\Gamma_\Sigma} \Phi^2 \; \overrightarrow{n}_0 \, \mathrm{d}\Gamma + \iint_\Sigma \left(\frac{1}{2}\, \nabla\Phi \cdot \nabla\Phi \; \overrightarrow{n} - \nabla\Phi \; \nabla\Phi \cdot \overrightarrow{n} \right) \mathrm{d}S =$$

$$-\frac{\omega^2}{2\,g} \int_{\Gamma_S} \Phi^2 \; \overrightarrow{n}_0 \, \mathrm{d}\Gamma + \iint_S \left(\frac{1}{2}\, \nabla\Phi \cdot \nabla\Phi \; \overrightarrow{n} - \nabla\Phi \; \nabla\Phi \cdot \overrightarrow{n} \right) \mathrm{d}S \tag{3}$$

[1]quoted from Landweber (1967)

where the normal vector $\overrightarrow{n}$ over S is now into the fluid.

Taken as a time-averaged value, either side of this equation is nothing but the horizontal drift force as obtained from momentum consideration. Taking Σ to infinity yields the Maruo expressions of the drift force. Equation (3) means that the control surface can also be taken coinciding with the body itself (Molin, 1979).

We now assume that the velocity potential Φ_P is generated by a source density σ over the wetted surface S. This induces a flow within the body as well, with velocity potential $\Phi_1 = \Phi_I + \Phi_{P1}$. For clarity we index with the subscript $_2$ the velocity potential outside the body: $\Phi_2 = \Phi$.

Identity (1) also holds inside the body:

$$\iint_{S\cup F_1} \left(\frac{1}{2} \nabla\Phi_1 \cdot \nabla\Phi_1 \; \overrightarrow{n} - \nabla\Phi_1 \; \nabla\Phi_1 \cdot \overrightarrow{n} \right) \mathrm{d}S \equiv \overrightarrow{0} \tag{4}$$

with F_1 the internal free surface. Restricting ourselves again to the horizontal components, we deduce

$$\iint_{S} \left(\frac{1}{2} \nabla\Phi_1 \cdot \nabla\Phi_1 \; \overrightarrow{n} - \nabla\Phi_1 \; \nabla\Phi_1 \cdot \overrightarrow{n} \right) \mathrm{d}S - \frac{\omega^2}{2\,g} \int_\Gamma \Phi_1^2 \; \overrightarrow{n}_0 \; \mathrm{d}\Gamma \equiv \overrightarrow{0}. \tag{5}$$

Substracting (5) from the right-hand side of equation (3), the drift force is obtained as

$$\begin{aligned} \overrightarrow{F}_d &= -\rho \frac{\omega^2}{2\,g} \int_{\Gamma_S} \left(\Phi_2^2 - \Phi_1^2\right) \; \overrightarrow{n}_0 \; \mathrm{d}\Gamma \\ &\quad + \rho \iint_S \left[\frac{1}{2} \left(\nabla\Phi_2 \cdot \nabla\Phi_2 - \nabla\Phi_1 \cdot \nabla\Phi_1\right) \; \overrightarrow{n} - \nabla\Phi_2 \; \nabla\Phi_2 \cdot \overrightarrow{n} + \nabla\Phi_1 \; \nabla\Phi_1 \cdot \overrightarrow{n} \right] \mathrm{d}S \end{aligned} \tag{6}$$

where time-averaged value is implied.

With a source singularity distribution, we have $\Phi_1 = \Phi_2$ over S and $\nabla\Phi_2 - \nabla\Phi_1 = \sigma \; \overrightarrow{n}$.

As a result the integral over the floatation contour Γ_S in (6) is zero identically. Replacing $\nabla\Phi_1$ with $\nabla\Phi_2 - \sigma \; \overrightarrow{n}$ in the integral over S, we obtain the simple expression:

$$\overrightarrow{F}_d = -\rho \iint_S \sigma \; \left(\nabla\Phi_2 - \frac{1}{2} \sigma \; \overrightarrow{n} \right) \mathrm{d}S. \tag{7}$$

The velocity potential Φ_2 being expressed as

$$\Phi_2 = \Phi_I + \frac{1}{4\pi} \iint_S \sigma(Q,t) \; G(P,Q) \; \mathrm{d}S_Q, \tag{8}$$

one has

$$\nabla\Phi_2 - \frac{1}{2} \sigma \; \overrightarrow{n} = \nabla\Phi_I + \frac{1}{4\pi} \iint_S \sigma(Q,t) \; \nabla_P G(P,Q) \; \mathrm{d}S_Q. \tag{9}$$

Further it appears that the Rankine term $1/PQ$ in the Green function $G(P,Q)$ does not contribute. This would take the form

$$\begin{aligned} \overrightarrow{F}_{dR} &= -\frac{\rho}{4\pi} \iint_S \sigma(P) \; \mathrm{d}S_P \iint_S \sigma(Q) \; \nabla_P \left(\frac{1}{PQ} \right) \mathrm{d}S_Q \\ &= -\frac{\rho}{4\pi} \iint_S \mathrm{d}S_P \iint_S \mathrm{d}S_Q \; \sigma(P) \; \sigma(Q) \; \frac{\overrightarrow{PQ}}{PQ^3} \\ &= \frac{\rho}{4\pi} \iint_S \mathrm{d}S_P \iint_S \mathrm{d}S_Q \; \sigma(P) \; \sigma(Q) \; \frac{\overrightarrow{QP}}{PQ^3} \\ &= -\overrightarrow{F}_{dR} \end{aligned}$$

So $\overrightarrow{F}_{dR} \equiv \overrightarrow{0}$ and, finally:

$$\overrightarrow{F}_d = -\rho < \iint_S \sigma \; \overrightarrow{V} \; \mathrm{d}S > \tag{10}$$

with

$$\overrightarrow{V} = \nabla\Phi_I + \frac{1}{4\pi} \iint_S \sigma(Q,t) \; \nabla_P H(P,Q) \; \mathrm{d}S_Q \tag{11}$$

and $G = 1/PQ + H(P,Q)$. In an unbounded fluid domain with imposed incident kinematics, $H \equiv 0$ and the usual Lagally expression is recovered.

It is straight-forward to establish that the associated expression for the drift moment in yaw is

$$M_{dz} = -\rho < \int\int_S \sigma \, \overrightarrow{OP} \wedge \overrightarrow{V} \, \mathrm{d}S > \cdot \overrightarrow{k} \tag{12}$$

3 Comments

Alike Maruo's and Newman's formulations, in the case of a surface piercing body, these expressions are restricted to the horizontal components. Extensions to the vertical components would involve a line integral over Γ_S (see Molin & Hairault, 1983) and an integral over the interior free surface. In the case of a fully submerged body there are no corrections and the vertical components are delivered as well (which is not true with the far-field method).

Another advantage over the classical far-field method is that individual drift forces are obtained in multiple body situations. Equation (7) directly applies to this case. The Rankine part can also be removed, but only the Rankine contribution of the considered body. All velocity components due to the presences of the other bodies must be added to the incoming wave velocity when computing (10). Numerical calculations have shown that there is no decisive accuracy gain in using equations (10) (11) in place of equation (7), which is much simpler to implement.

Advantages over the pressure integration method is that the obtained expressions are less prone to numerical inaccuracies, especially in the case of a body with sharp corners.

4 Results

First we consider two spheres half-immersed with a separation distance, from center to center, equal to 3 times their radius a. The waterdepth is infinite. The spheres are free to respond to the waves, which propagate in a direction perpendicular to the arrangement. Figure 1 shows the obtained drift force in x (wave direction) and y, for the sphere located in $x = 0$, $y = 1.5\,a$. The pressure integration method and the Lagally formulation are used, with the Lagally results being obtained from equation (7) (without removing the Rankine part), implemented in the ECN code Aquadyn. For the x-component, the far-field result is also given (by symmetry it is half the force on the total arrangement). It can be seen that, except in the vicinity of the first irregular frequency, the agreement between the different formulations is excellent. The transverse force F_{dy} is mostly positive, meaning a repulsive force from the other sphere. It can also be observed that the peak value of the in-line drift force is much larger than for an isolated sphere (Kudou, 1977).

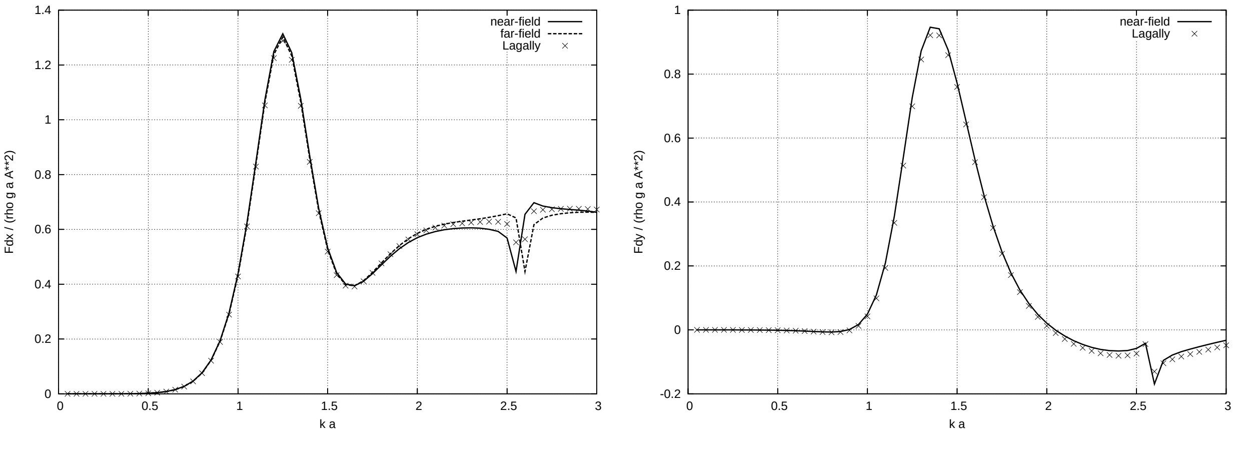

Figure 1: Normalized drift force acting on one sphere in a twin arrangement. x- (left) and y- (right) components.

Next we consider the two ships case studied by Kashiwagi (2004) — see also Kashiwagi *et al.*, 2005. One ship is a modified Wigley hull model (ship A) while the other is a rectangular barge model (ship B). They are both 2 m in length with a separation distance S equal to 1.797 m. The waves are coming from abeam with ship A on the weather side of the arrangement, held motionless.

In figure 2 we present the normalized drift forces in sway, as obtained with the Lagally formulation, and as calculated by Kashiwagi (2004) with the pressure integration method (and a higher-order boundary element method — HOBEM). The results are plotted vs. the wavelength λ divided by the length L of the two ships. It can be seen that the agreement is excellent, except at small λ/L values where our meshing becomes too coarse.

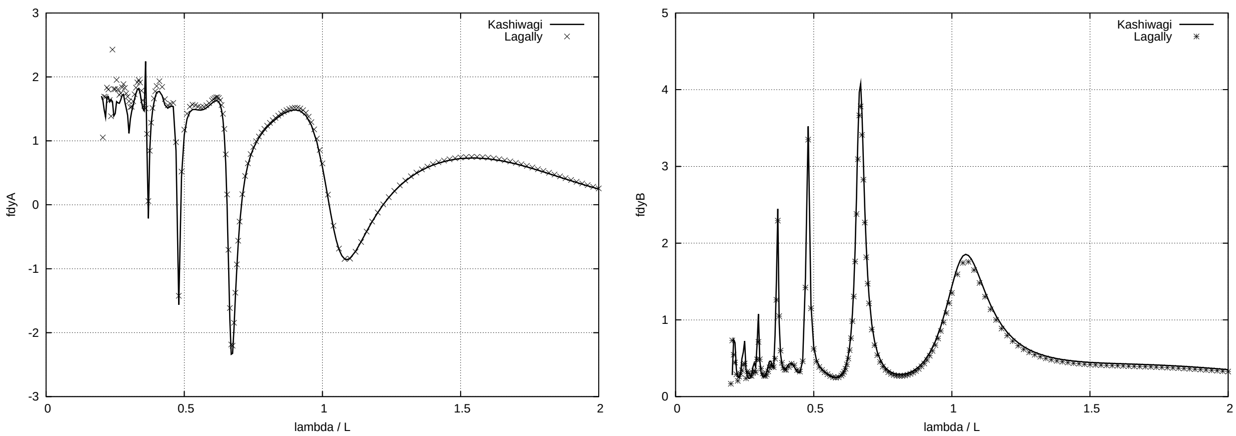

Figure 2: Normalized sway drift forces on ships A (left) and B (right). Comparison between the Lagally formulation and the near-field results of Kashiwagi.

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Discusser - B Chen:

My comments are to clarify the major differences between the middle-field formulation (MF) in Chen (2006, JEM) and the Lagally formulation (LF) presented here. Derived from the pressure-integration formulation and written on control surfaces at a distance from the surrounded body by using two variants of Stokes' theorem, MF is as general as the pressure-integration formulation (6 DOF, Multi-bodies, free of irregular frequencies, drift load and QTF) and as accurate as the momentum formulation. LF is interesting, although limited to 3 DOF drift components and with effect of irregular frequencies, if the speculation to extend it to the computation of QTF with better precision than the pressure-integration becomes true.

Reply:

None

Discusser - Duan Wen Yang:

The presented formula (10)-(12) is very interesting, it confirms the results of our research presented in a Chinese book written by Dai (1998) and in his notes (2005) on the application for multi-body interaction. The results for floating bodies still have the influence of irregular frequencies, this is the difficulties we encountered, your suggestion to overcome this difficulty is welcome.

Reply:

You should have this book by Dai translated into English.

I have not looked in detail at the problem of irregular frequencies but it is obvious that the usual tricks that are being applied to remove them (lid over the interior free surface) would affect the calculated drift force. I agree that this would be a drawback of the Lagally formulation.

Discusser - T. Miloh:

For the steady Lagally theorem, in the case where the body can be represented by a distribution of sources (continuity of the potential across the boundary), which can be a line, surface or volume distribution the force can be expressed in terms of a corresponding line, surface or volume integral involving the product of the source strength times the induced velocity, excluding the self induced (Rankine part). Thus, eqs (10) + (11) are straight manifestation of the Lagally theorem and need not be proven. These can be easily extended for the unsteady case by adding the term
$-\frac{\partial \rho}{\partial t} < \iint_S \sigma \overline{r}\, \mathrm{d}S >$ where $\overline{r}$ is the radius verctor of the source distribution.

Reply:

All the papers I have found on Lagally theorem deal with unbounded fluid domains, and use the fact that some integrals at infinity are nil. It was not obvious to me how Lagally theorem had to be modified in the case of multiple bodies with a free surface.