# Oblique wave scattering by a circular cylinder in two-layer fluid with an ice-cover

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## 1. Introduction

Linton and McIver (LM)(1995) developed the general theory for two-dimensional wave motion in a two-layer fluid in which the lower fluid extends infinitely downwards while the upper fluid has a free surface. They also considered interaction of waves with a horizontal cylinder submerged in either of the two layers. This problem arose from modelling an underwater pipe bridge across Norwegian fjords consisting of a layer of fresh water on the top of a deep layer of salt water. Linton and Cadby (LC) (2002) extended the work of LM to oblique scattering. During winter the fjords are covered by ice, and this has motivated us to extend the problem of LC to an ice-covered two-layer fluid wherein the ice-cover is modelled as a thin elastic plate. The reflection and transmission coefficients for oblique scattering by a horizontal circular cylinder submerged in the lower fluid are obtained and depicted graphically against the wave number for various values of the angle of incidence in a number of figures. It is noted that for normal incidence, the reflection coefficients are identically zero.

## 2. Mathematical formulation

A cartesian co-ordinate system is chosen in which the $y$-axis points vertically upwards with $y=0$ as the mean interface and $y=h>0$ as the mean position of ice-cover. For oblique waves the velocity potential has the form $\Phi(x,y,z,t)=Re\{\phi(x,y)e^{-i\sigma t+i\gamma z}\}$. The potential functions $\phi^I$ and $\phi^{II}$ in the upper and lower layers respectively satisfy the modified Helmholtz equation

$$(\nabla^2-\gamma^2)\phi^{I,II}=0, \qquad \text{in appropriate layers.} \tag{2.1}$$

The linearized boundary conditions at the interface are

$$\phi^I_y=\phi^{II}_y \qquad \text{on } \ y=0, \tag{2.2}$$

$$s(\phi^I_y-K\phi^I)=\phi^{II}_y-K\phi^{II} \quad \text{on} \quad y=0 \tag{2.3}$$

where $s=\frac{\rho_1}{\rho_2}(<1)$, $\rho_1$ being the density of the upper layer while $\rho_2$ that of the lower fluid, and at the ice-cover, is

$$\{D(\frac{\partial^2}{\partial x^2}-\gamma^2)^2+1-\epsilon K\}\phi^I_y-K\phi^I=0 \text{ on } y=h, \tag{2.4}$$

where $K=\frac{\sigma^2}{g}$, $D$ is the flexural rigidity of the material of ice-cover in the usual notation, and

$$\nabla\phi^{II}\to 0 \qquad \text{as } \ y\to-\infty. \tag{2.5}$$

In a two-layer fluid progressive waves have the form (except for multiplicative constant)

$$\phi^I=e^{\pm ix(k^2-\gamma^2)^{\frac{1}{2}}}\left(V_1(k)e^{ky}+V_2(k)e^{-ky}\right), \tag{2.6}$$

$$\phi^{II}=e^{\pm ix(k^2-\gamma^2)^{\frac{1}{2}}}e^{ky}\left(V_1(k)-V_2(k)\right), \tag{2.7}$$

where $k$ satisfies the dispersion relation

$$G(k)\equiv V_3(k)+V_4(k)=0, \tag{2.8}$$

where $V_1(k),V_2(k)=(U(k)\pm K)e^{\mp kh}$, $V_3(k)=-\left(k(1-s)-K\right)\left(U(k)\sinh kh-K\cosh kh\right)$, $V_4(k)=Ks\left(U(k)\cosh kh-K\sinh kh\right)$, $U(k)=k(Dk^4+1-\epsilon K)$. This dispersion equation has exactly two positive real roots $\lambda_1$ and $\lambda_2(\lambda_1<\lambda_2)$, say.

For the case $k=\lambda_j(j=1,2)$ progressive waves are thus of the form

$$\phi^I=e^{\pm i\beta_j x}g_j(y), \qquad \phi^{II}=e^{\pm i\beta_j x+\lambda_j y}, \tag{2.9}$$

where $\beta_j=(\lambda_j^2-\gamma^2)^{\frac{1}{2}}, j=(1,2)(\beta_j=\lambda_j$ for $\gamma=0)$ and

$$g_j(y)=\frac{\{\lambda_j(1-s)-K\}\left(V_1(\lambda_j)e^{\lambda_j y}+V_2(\lambda_j)e^{-\lambda_j y}\right)}{V_4(\lambda_j)}. \tag{2.10}$$

In any wave scattering problem therefore, the far-field will take the form of incoming and outgoing waves at each of the wave numbers $\lambda_j(j=1,2)$. It is given by

$$\phi^{I,II}\sim\left(A^\pm e^{\pm i\beta_1 x}+C^\pm e^{\mp i\beta_1 x}\right)\left(g_1(y),e^{\lambda_1 y}\right)$$
$$+\left(B^\pm e^{\pm i\beta_2 x}+D^\pm e^{\mp i\beta_2 x}\right)\left(g_2(y),e^{\lambda_2 y}\right), \tag{2.11}$$

as $x\longrightarrow\pm\infty$, for which in the notation of LM

$$\phi\sim\left(A^-,B^-,C^-,D^-;A^+,B^+,C^+,D^+\right).$$

An incident plane wave $\phi_{\text{inc}}$ of wave number $\lambda_1$ has the form

$$\phi^I_{\text{inc}}=e^{i\lambda_1 x\cos\alpha}g_1(y), \phi^{II}_{\text{inc}}=e^{i\lambda_1 x\cos\alpha+\lambda_1 y}. \tag{2.12}$$

In this case $\gamma=\lambda_1\sin\alpha$, $\beta_1=\lambda_1\cos\alpha$, $\beta_2=(\lambda_2^2-\lambda_1^2\sin^2\alpha)^{\frac{1}{2}}$. We know that $\beta_2$ is real since $\lambda_2>\lambda_1$ and thus waves of wave number $\lambda_2$ will exist for all values of $\lambda_1$ and $\alpha$.

An incident plane wave of wave number $\lambda_2$ is given by

$$\phi^I_{\text{inc}}=e^{i\lambda_2 x\cos\alpha}g_2(y), \phi^{II}_{\text{inc}}=e^{i\lambda_2 x\cos\alpha+\lambda_2 y}. \tag{2.13}$$

In this case $\gamma = \lambda_2 \sin\alpha$, $\beta_1 = (\lambda_1^2 - \lambda_2^2 \sin^2\alpha)^{\frac{1}{2}}$, $\beta_2 = \lambda_2 \cos\alpha$. For a given angle $\alpha$ there may be a value of $K$ for which $\lambda_1 = \lambda_2 \sin\alpha$ and thus $\beta_1 = 0$. This value of $K$ is called cut-off frequency and denoted by $K_c$ in the notation of LC. For some values of $K$ for which $\lambda_1 < \lambda_2 \sin\alpha$ (for fixed $\alpha$) $\beta_1$ becomes imaginary and in that case there exists no propagating wave of wave number $\lambda_1$. Fig.1 shows the cut-off frequency $K_c a$, plotted against incident wave angle $\alpha = \sin^{-1}(\lambda_1/\lambda_2)$, for $s = 0.5$, $h/a = 2$, $\epsilon/a = 0.01$, $a$ being the radius of the submerged cylinder of §3. The different curves correspond to $D/a^4 = 2, 1.5, 1, 0.5, 0.1$. It is observed from this figure that for any angle $\alpha$ situated on the right side of the curve there are no propagating waves of wave number $\lambda_1$ for any frequency. It may be noted that for very small $\frac{D}{a^4}$ and $\frac{\epsilon}{a}$ the $K_c a$ curve has almost the same features of the curve given in fig.1 of LC.

## 3. Cylinder in the lower fluid

Let a horizontal circular cylinder of radius $a$ have its axis at $y = f(< 0)$, and polar coordinates $(r, \theta)$ be defined in the $(x, y)$-plane by $x = r\sin\theta$ and $y = f - r\cos\theta$. Symmetric and antisymmetric multipoles $\phi_n^s$ $(\geq 0)$ and $\phi_n^a$ $(\geq 1)$ respectively are defined by (in the notation of LC)

$$\phi_n^{Is} = (-1)^n \int_0^\infty \Theta_1(k) \left(A(k)e^{vy} + B(k)e^{-vy}\right) dk, \tag{3.1}$$

$$\phi_n^{IIs} = K_n(\gamma r)\cos n\theta + (-1)^n \int_0^\infty \Theta_1(k) C(k) e^{vy} dk, \tag{3.2}$$

$$\phi_n^{Ia} = (-1)^{n+1} \int_0^\infty \Theta_2(k) \left(A(k)e^{vy} + B(k)e^{-vy}\right) dk, \tag{3.3}$$

$$\phi_n^{IIa} = K_n(\gamma r)\sin n\theta + (-1)^{n+1} \int_0^\infty \Theta_2(k) C(k) e^{vy} dk, \tag{3.4}$$

where $\Theta_1(k) = \cosh nk \cos(\gamma x \sinh k)$, $\Theta_2(k) = \sinh nk \sin(\gamma x \sinh k)$, $v = \gamma \cosh k$ and

$$A(k), B(k) = (KV_1(v), KV_2(v))\, e^{vf}/G(v), \tag{3.5}$$

$$C(k) = (V_3(v) + V_4(v))\, e^{vf}/G(v). \tag{3.6}$$

The path of integration in the integrals in (3.1) to (3.4) is indented below the poles at $k = \mu_j$, where $\gamma \cosh\mu_j = \lambda_j, j = 1, 2$. The far-field form of the multipoles are given by

$$\phi_n^{IIs} \sim (-1)^n \pi i \left(C^{\mu_1} \cosh n\mu_1 e^{\pm i\beta_1 x + \lambda_1 y} + C^{\mu_2} \cosh n\mu_2 e^{\pm i\beta_2 x + \lambda_2 y}\right), \tag{3.7}$$

$$\phi_n^{IIa} \sim \mp(-1)^n \pi \left(C^{\mu_1} \sinh n\mu_1 e^{\pm i\beta_1 x + \lambda_1 y} + C^{\mu_2} \sinh n\mu_2 e^{\pm i\beta_2 x + \lambda_2 y}\right), \tag{3.8}$$

as $x \to \pm\infty$. Here $C^{\mu_j}$ is the residue of $C(k)$ at $k = \mu_j (j = 1, 2)$ given by

$$C^{\mu_j} = (V_3(\lambda_j) + V_4(\lambda_j))\, e^{\lambda_j f}/\beta_j G'(\lambda_j). \tag{3.9}$$

Using the well-known result

$$e^{\frac{1}{2}X(T+T^{-1})} = \sum_{m=0}^\infty \frac{1}{2}\epsilon_m \left(T^m + T^{-m}\right) I_m(X), \tag{3.10}$$

where $\epsilon_0 = 1, \epsilon_m = 2, m \geq 1$ and $I_m(X)$ is the modified Bessel function of first kind, (3.2) and (3.4) can be expanded in terms of polar co-ordinates as

$$\phi_n^{IIs} = K_n(\gamma r)\cos n\theta + \sum_{m=0}^\infty A_{nm}^{(s)} I_m(\gamma r)\cos m\theta, \tag{3.11}$$

$$\phi_n^{IIa} = K_n(\gamma r)\sin n\theta + \sum_{m=1}^\infty A_{nm}^{(a)} I_m(\gamma r)\sin m\theta, \tag{3.12}$$

where

$$A_{nm}^{(s)} = \epsilon_m (-1)^{n+m} \int_0^\infty e^{vf} \cosh mk \cosh nk C(k) dk, \tag{3.13}$$

$$A_{nm}^{(a)} = 2(-1)^{n+m} \int_0^\infty e^{vf} \sinh mk \sinh nk C(k) dk. \tag{3.14}$$

*Incident wave number* $\lambda_1$

Let us consider the case of an incident plane wave of wave number $\lambda_1$ making an angle $\alpha$ with the positive $x$-axis, so that $\gamma = \lambda_1 \sin\alpha$. The incident wave potential (2.12) has the form

$$\phi_{\text{inc}}^{II} = e^{\lambda_1 f} \sum_{m=0}^\infty \epsilon_m (-1)^m I_m(\gamma r) \left(\cosh m\nu \cos m\theta - i \sinh m\nu \sin m\theta\right) \tag{3.15}$$

where $\cosh\nu = \lambda_1/\gamma = 1/\sin\alpha$. We write the resulting velocity potential as

$$\phi_{\lambda_1} = \phi_{\text{inc}} + \sum_{n=0}^\infty \left(a_n \phi_n^a + b_n \phi_n^s\right), \tag{3.16}$$

where $a_n$ and $b_n$ are unknown coefficients. To solve for $a_n$ and $b_n$ we substitute (3.11),(3.12) and (3.15) into (3.16) and apply the body boundary condition $\frac{\partial \phi_{\lambda_1}}{\partial r} = 0$ on $r = a$. We obtain two infinite systems of linear equations for these unknowns as given by

$$\frac{a_m}{Z_m} + \sum_{n=1}^\infty A_{nm}^{(a)} a_n = 2i(-1)^m e^{\lambda_1 f} \sinh m\nu, \quad m = 1, 2, \cdots \tag{3.17}$$

$$\frac{b_m}{Z_m} + \sum_{n=0}^\infty A_{nm}^{(s)} b_n = (-1)^{m+1} \epsilon_m e^{\lambda_1 f} \cosh m\nu, \quad m = 0, 1, \cdots \tag{3.18}$$

where $Z_m = I'_m(\gamma r)/K'_m(\gamma r)$. The far-field form for $\phi_{\lambda_1}$, in the lower fluid layer, can be written as

$$\phi_{\lambda_1}^{II} \sim \begin{cases} e^{i\beta_1 x+\lambda_1 y} + R_{\lambda_1} e^{-i\beta_1 x+\lambda_1 y} + r_{\lambda_1} e^{-i\beta_2 x+\lambda_2 y} \\ \qquad\qquad\qquad\qquad \text{as } x \to -\infty, \\ T_{\lambda_1} e^{i\beta_1 x+\lambda_1 y} + t_{\lambda_1} e^{i\beta_2 x+\lambda_2 y} \\ \qquad\qquad\qquad\qquad \text{as } x \to \infty, \end{cases} \tag{3.19}$$

where $R_{\lambda_1}, r_{\lambda_1}$ are the reflection coefficients for reflected waves of wave numbers $\lambda_1$ and $\lambda_2$ respectively due to an incident wave of wave number $\lambda_1$, and similarly for transmission coefficients $T_{\lambda_1}$ and $t_{\lambda_1}$. Using (3.16), (3.7), (3.8) the reflection and transmission coefficients are obtained as

$$R_{\lambda_1}, r_{\lambda_1} = \pi C^{\mu_{1,2}} \sum_{m=0}^{\infty} (-1)^m P_{1,2}, \tag{3.20}$$

$$T_{\lambda_1} - 1, t_{\lambda_1} = \pi C^{\mu_{1,2}} \sum_{m=0}^{\infty} (-1)^m Q_{1,2}, \tag{3.21}$$

where $P_{1,2}, Q_{1,2} = i b_m \cosh m\mu_{1,2} \pm a_m \sinh m\mu_{1,2}$.

*Incident wave number* $\lambda_2$

We consider the case of an incident plane wave of wave number $\lambda_2$. The expression of incident wave potential is the same as (3.15), except that $\lambda_1$ is replaced by $\lambda_2$. The velocity potential $\phi_{\lambda_2}$ for this problem can again be expanded in multipoles similar to (3.16) and the equations for $a_n$ and $b_n$ are similar to (3.17) and (3.18) with $\lambda_1$ replaced by $\lambda_2$. Using the far-field forms of the multipoles given by (3.7) and (3.8) in $\phi_{\lambda_2}$ we find that the expressions for reflection coefficients $R_{\lambda_2}$ and $r_{\lambda_2}$ are similar to (3.20) with appropriate changes, and the transmission coefficients are given by

$$T_{\lambda_1}, t_{\lambda_1} - 1 = \pi C^{\mu_{1,2}} \sum_{m=0}^{\infty} (-1)^m Q_{1,2}. \tag{3.22}$$

For the cylinder in the upper layer, the expressions for reflection and transmission coefficients have also been obtained but are not presented here.

## 3. Discussion

In figures 2-5 the reflection and transmission coefficients are shown for the case of wave of wave number $\lambda_1$ incident on the cylinder in the lower fluid for $\frac{\epsilon}{a} = 0.01, \frac{h}{a} = 2, s = 0.5, \frac{f}{a} = -2, \frac{D}{a^4} = 1.5$. From figures 2 and 4 it is observed that as the angle of incidence increases, $|R_{\lambda_1}|$ increases while $|T_{\lambda_1}|$ decreases. Also $|R_{\lambda_1}|$ is somewhat small in comparison to that of LC and $|T_{\lambda_1}|$ is somewhat large in comparison to that of LC. $|r_{\lambda_1}|$ and $|t_{\lambda_1}|$ of the waves of wave number $\lambda_2$, shown in figures 3 and 5, are small in comparison to those of $\lambda_1$. The case of an incident wave of wave number $\lambda_2$ is more interesting due to the presence of cut-off frequency. For this case, figures 6-9 show $|R_{\lambda_2}|$, $|r_{\lambda_2}|$, $|T_{\lambda_2}|$, $|t_{\lambda_2}|$ against $Ka$. When $\alpha = 0.335$, which is greater than the critical angle $\alpha = 0.3307$ for the given values of the different parameter, there are no waves of wave number $\lambda_1$ propagating in the fluid. From fig.1 we have the following cut-off frequencies: $K_c a = 0.07; (0.09, 0.86); (0.13, 0.665); (0.17, 0.54)$ corresponding to the incident angles 0.24, 0.26, 0.29, 0.31 respectively. For $\alpha = 0.24$, only for frequencies greater than the cut-off frequency will there be conversion of wave from one mode to the other but for other angles, only for frequencies lying between two cut-off frequencies will there be energy conversion.

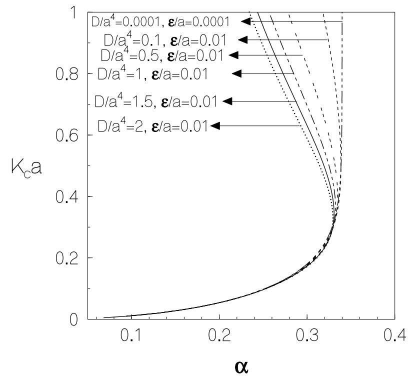


Fig.1: Cut-off frequency $K_c a$ due to an incident wave of wave number $\lambda_2$: (h/a=2, s=0.5 )

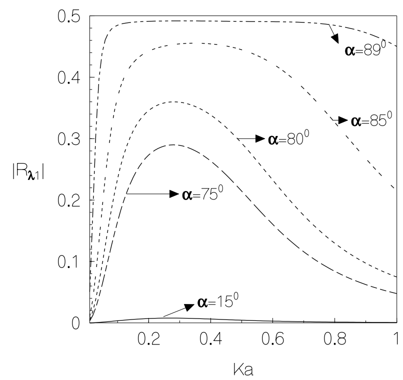


fig.2: Reflection coefficient due to a wave of wavenumber $\lambda_1$ incident on a cylinder in the lower layer: ($D/a^4$=1.5, ε=.01, h/a=2, s=.5, f/a=-2)

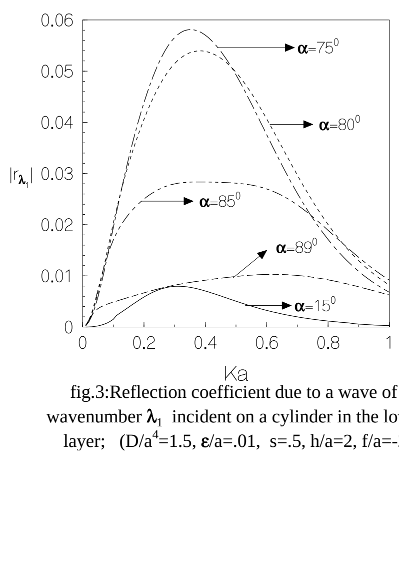


fig.3:Reflection coefficient due to a wave of wavenumber $\lambda_1$ incident on a cylinder in the lower layer; ($D/a^4$=1.5, ε/a=.01, s=.5, h/a=2, f/a=-2)

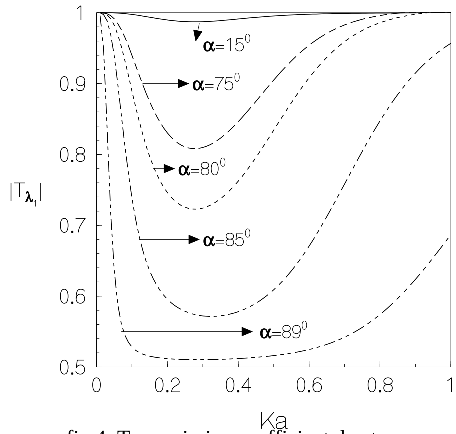


fig.4: Transmission coefficient due to a wave of wavenumber $\lambda_1$ incident on a cylinder in the lower layer; ($D/a^4$=1.5, $\epsilon/a$=.01, h/a=2, s=.5, f/a=-2)

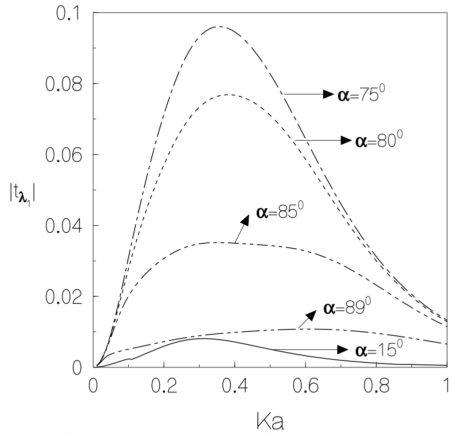


fig.5: Transmission coefficient due to a wave of wavenumber $\lambda_1$ incident on a cylinder in the lower layer;($D/a^4$=1.5, $\epsilon/a$=.01, s=.5, h/a=2, f/a=-2 )

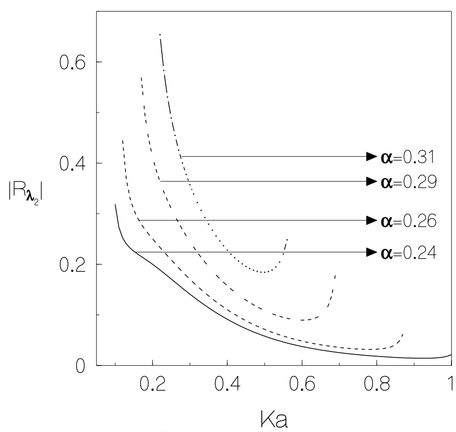


Fig.6: Reflection coefficient due to a wave of wavenumber $\lambda_2$ incident on a cylinder in the lower layer;($D/a^4$=1.5, $\epsilon/a$=0.01, h/a=2, s=0.5, f/a=-2 )

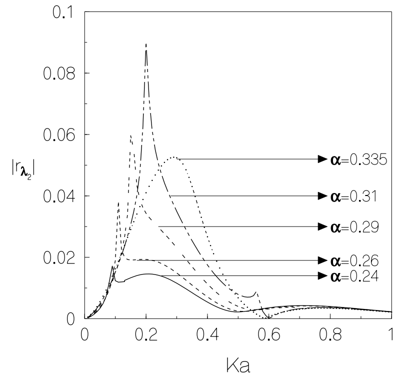


Fig.7: Reflection coefficient due to a wave of wave number $\lambda_2$ incident on a cylinder in the lower layer; ($D/a^4$=1.5, $\epsilon/a$=0.01, h/a=2, s=0.5, f/a=-2)

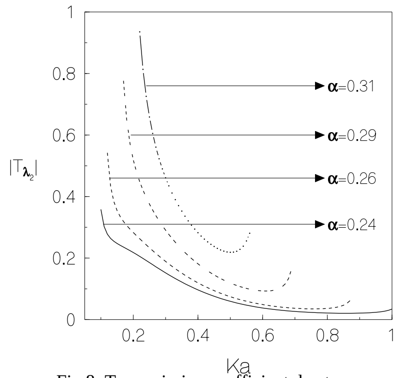


Fig.8: Transmission coefficient due to a wave of wavenumber $\lambda_2$ incident on a cylinder in the lower layer;($D/a^4$=1.5, $\epsilon/a$=0.01,s=0.5, h/a=2, f/a=-2)

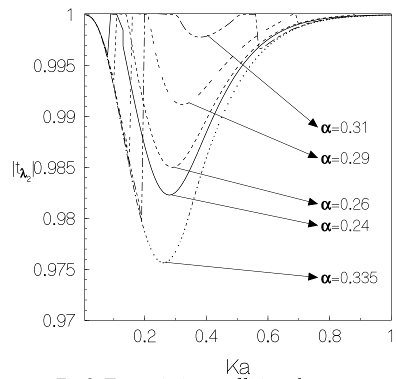


Fig.9: Transmission coefficient due to a wave of wavenumber $\lambda_2$ incident on a cylinder in the lower layer; ($D/a^4$=1.5, $\epsilon/a$=0.01,h/a=2, s=0.5, f/a=-2 )

## Acknowledgement

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## References

1. Linton,C. M. and McIver, M., *J. Fluid Mech.* **304**, (1995) 213-229.

2. Linton, C. M. and Cadby, J. R.,*J. Fluid Mech.* **461**,(2002) 343-364.