Lifting Surfaces with Circular Planforms

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Abstract

Wings with circular planforms are not common in aeronautics, although there are disc-like projectiles with sporting and recreational applications. There are some semi-analytic approaches in the literature (including by Miloh) to solution for the flow over thin circular lifting surfaces, which yield exact or nearly exact results for flat discs at small angle of attack. These solutions are reviewed and compared with direct numerical solutions of the lifting surface integral equation. Solutions for non-flat discs with small twist or camber are also discussed, including choice of twist or camber to minimise induced drag, or to achieve a favourable placement of the centre of pressure. Axisymmetrically cambered discs at zero angle of attack are paid special attention.

Introduction

The loading $\Delta p(x,y)$ on a thin lifting surface $z = f(x,y)$ with a planform B in an x-directed stream U satisfies the lifting surface integral equation (LSIE)

$$\iint_B \Delta p(\xi,\eta) W(x-\xi, y-\eta) d\xi d\eta = 4\pi\rho U^2 f_x(x,y)$$

where

$$W(x,y) = \frac{1}{y^2}\left[1 + \frac{x}{\sqrt{x^2+y^2}}\right]$$

is the downwash induced by a unit horse-shoe vortex at the origin. The LSIE needs to be solved subject to a Kutta condition that Δp vanish at the trailing edge of B. Reasonably (but not outstandingly) accurate numerical codes are available to solve the LSIE for given planforms B and mean surface shapes $f(x,y)$. The basic outputs are lift coefficient C_L and pitching moment via the centre of pressure location $x = x_P$, and we use a code (called here "TS", for Tuck and Standingford 1997) which gives these quantities with about three figure accuracy, the planform B being discretised into 5000 rectangular panels.

On the other hand, for the special case where B is the unit circle

$$x^2 + y^2 \leq 1$$

it is possible to construct much more accurate solutions, in the form of truncated infinite series. This is best seen indirectly, by noting that our real task is to solve the 3D Laplace equation exterior to a circular disc which is the limiting form of the oblate spheroid

$$z = \pm\epsilon\sqrt{1 - x^2 - y^2}$$

as $\epsilon \to 0$. Hence we can write $\Delta p(x, y)$ as the value on $z = 0$ of an infinite series whose terms are fundamental solutions of Laplace's equation in oblate spheroidal polar coordinates, which involve Legendre functions.

Solutions of this series type have been obtained by Robinson and Laurmann (1956), Jordan (1973), Hauptman and Miloh (1986), Boersma (1989), and others, with various assumptions made about the mean surface function $f(x, y)$.

Flat discs at unit angle of attack

For the case $f(x, y) = -x$, very accurate results have been obtained by Jordan (1973) and confirmed by Boersma (1989). The lift coefficient is $C_L = 1.79002$ and the centre of pressure is at $x_P = -0.52086$, all 5 decimal places of accuracy (and more) being reliable. By contrast, the lift coefficient predicted by lifting-line theory is $C_L = 2.444$, and the 2D "quarter-chord" centre of pressure is $x_P = -0.5$. These results are inaccurate because they are valid only at high aspect ratio, and a circle is a low-aspect-ratio wing. Our general purpose LSIE solver TS produces $C_L = 1.79078$ (error 0.04%) and $x_P = -0.52194$ (error 0.2%).

Hauptman-Miloh twisted discs

Hauptman and Miloh (1986) constructed a simplified series solution which nevertheless gives a very accurate approximation to the above solution for a flat disc at unit angle of attack. In fact (Boersma 1989) it is the exact solution for a "twisted" disc

$$f(x, y) = -x + xg(y)$$

where $g(y)$ is a relatively small quantity. This distortion of the disc is so small that the Hauptman-Miloh lift coefficient $C_L = 1.79075$ is within 0.04% of the true flat-plate value. However, the centre of pressure at $x_P = -.52360$ is not quite so close to the flat-plate value, differing by 0.5%. In fact these values are almost as close to the flat-plate values as are the numerical results for the flat plate computed by the TS program. At other angles of attack, we just have to scale this solution proportionally, but note that then the twist also scales as the angle of attack changes.

Twisted discs with elliptic loading

Robinson and Laurmann (1956) also produced series solutions for a nearly-flat disc at unit angle of attack. However, they used a constraint that the chordwise-integrated loading vary exactly elliptically across the span. This means that the resulting wing is optimal from the point of view of minimisation of induced drag. Again this constraint results in a small twist $g(y)$, different from and somewhat larger than that of the Hauptman-Miloh disc. Again, the twist must vary in proportion to angle of attack, and this effect is more significant for the Robinson-Laurmann twist than for the Hauptman-Miloh twist.

Axisymmetrically cambered discs

Twist as introduced in the previous sections disturbs the axisymmetry of the circular planform. An alternative departure from a flat disc at zero angle of attack has

$$f(x, y) = g(r)$$

for some shape function $g(r)$, where $r = \sqrt{x^2 + y^2}$, maintaining axisymmetry.

For definiteness let us normalise so $g(0) = 0$ and $g(1) = -1$. The paraboloid of revolution $g(r) = -r^2 = -x^2 - y^2$ is of particular interest, noting that since only the longitudinal slope contributes to the linearised aerodynamics, the term "$-y^2$" can be ignored. Hence this gives the same lift and moment as for a circular disc with parabolic x-wise camber $f(x, y) = -x^2$, which has been studied by previous authors. Boersma (1989) gives $C_L = 1.86469$ and $x_P = 0.47064$. The values computed by TS are $C_L = 1.86842$ (error 0.2%) and $x_P = 0.47102$ (error 0.08%).

More generally, consider the "monomial" family of shapes $g(r) = -r^n$ for some power n. The above paraboloid is the case $n = 2$. As n increases, the slope of the body is more and more concentrated near its rim $r = 1$. Table 1 gives C_L and x_P for some members of this family. Note how little the centre of pressure changes within this family, staying near to $x_P \approx 0.47$ for all n. This seems characteristic of most convex axisymmetric bodies.

Power n	Lift coefficient C_L	Centre of pressure x_P
2	1.86842	0.47102
4	2.99223	0.47092
10	5.19459	0.47062
20	7.62819	0.47012

Axisymmetric discs with zero pitching moment.

Although concave-down axisymmetric discs at zero angle of attack tend to have negative (leading-edge down) pitching moments, it is easy by a linear combination of two or more such shapes (with at least one "upside-down") to eliminate the pitching moment entirely, so moving the centre of pressure to the axis. This is a

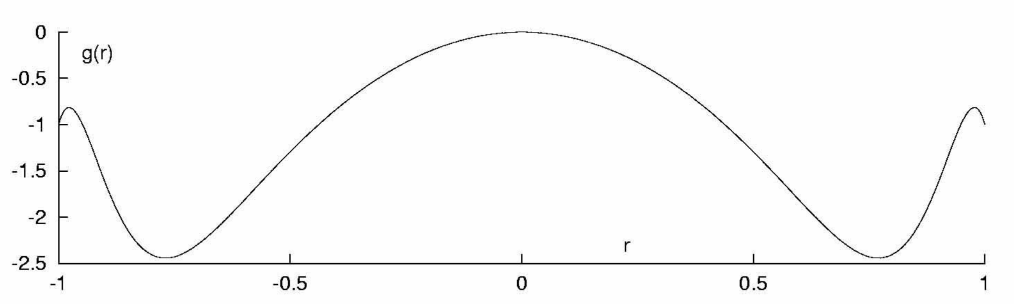

Figure 1: An axisymmetric disc with zero pitching moment and positive lift.

highly desirable property for any spinning device, as it eliminates gyroscopic forces inducing roll.

Unfortunately, since as we have seen, convex shapes also tend to have centres of pressure that are very close to each other, elimination of pitching moment in this way also almost eliminates lift! However, some apparent success retaining a small positive lift can be achieved by a linear combination of three monomial shapes. For example

$$g(r) = -5.21r^2 + 9.21r^{10} - 5.00r^{20}$$

has $C_L = 0.03322$ and $x_P = 0.01535$, so the pitching moment coefficient $C_L.x_P$ is less than 0.001. Figure 1 shows this (non-convex) shape.

References

BOERSMA, J. "Note on the lifting-surface problem for a circular wing in incompressible flow", *Quarterly Journal of Mechanics and Applied Mathematics* **42** (1989) 55–64.

HAUPTMANN, A. and MILOH, T. "On the exact solution of the linearized lifting-surface problem of an elliptic wing", *Quarterly Journal of Mechanics and Applied Mathematics* **39** (1986) 41–66.

JORDAN, P.F. "Exact solutions for lifting surfaces", *AIAA journal* **11** (1973) 1123–1129.

ROBINSON, A. and LAURMANN, J.A., *Wing theory*, Cambridge University Press, 1956.

TUCK, E.O. and STANDINGFORD, D.W.F., "Numerical solution of the lifting surface equation", *Boundary Element Technology XII*, Knoxville, Tenn, 9–11 April 1997. Proceedings ed. J.I. Frankel, C.A. Brebbia and M.A.H. Aliabadi, Computational Mechanics Publications, Southampton, pp. 195–206.