Second-order diffraction in short waves

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1 Introduction

Second-order wave loads are significant for various types of offshore structures. Many analytical studies have been performed, based on the assumption of potential flow and using perturbation expansions including all contributions of second order. Most results are for monochromatic incident waves, where the second-order effects include a second-harmonic and a time-independent component. Additional biharmonic effects must be considered in a spectrum, including both sums and differences of the first-order frequencies. Analytical studies are restricted either to two dimensions or to axisymmetric three-dimensional structures. Several numerical codes have been developed based on the panel method, to predict the second-order wave effects on more general structures in three dimensions.

When the incident waves are short, or the frequency ω is large, the first-order wave field is confined to a thin layer near the free surface. The first-order loads are asymptotically small, and can be analyzed by the method of geometric optics or ray theory. Little is known regarding second-order effects in this regime except for a few specific bodies. In general the second-order loads do not tend to zero (when normalized in the usual manner based on the square of the incident-wave amplitude). The simplest example is the oscillatory 'runup' at the waterline, which causes a concentrated pressure force equal to $\frac{1}{2}\rho g\zeta^2$, where ρ is the fluid density, g is the gravitational acceleration, and ζ is the first-order local free-surface elevation.

The pressure due to the second-order component of the potential is a more complicated cause of second-harmonic loads. Unlike the first-order diffraction field, the second-order potential persists at large depths below the free surface even when the first-order waves are very short (Newman, 1990). For specific bodies it has been shown that the second-harmonic force increases without bound, in proportion to the wavenumber ω^2/g in two dimensions (Wu & Eatock Taylor, 1989; McIver, 1994) and in proportion to the frequency ω for a vertical cylinder in three dimensions (Newman, 1996). Less is known regarding the behavior of the sum- and difference-frequency components in short biharmonic waves.

Our objective here is to derive asymptotic results for the second-order potential and loads in the short-wavelength regime. A simplified problem is considered, where waves are incident upon a two-dimensional (cylindrical) body which intersects the free surface. Both oblique and normal incidence are considered. For three-dimensional applications the results can be applied to elongated vessels using a strip-theory synthesis, and also to compact bodies where the radius of curvature along the waterline is large compared to the wavelength. Since the body motions are small in short waves, it is reasonable to consider only the diffraction problem where the body is fixed in position. The fluid depth is assumed to be infinite.

2 Formulation

The horizontal coordinates x, y are in the plane of the free surface and the z-axis is positive upward. The body is cylindrical with its generators parallel to the y-axis. The sides intersect the free surface vertically along the waterlines $x = \pm b$. Two incident waves ($i = 1, 2$) are considered, with complex amplitudes A_i, frequencies ω_i, wavenumbers $k_i = \omega_i^2/g$ and incidence angles β_i relative to the positive x-axis. The (x, y) components of the wavenumber vector are $u_i = k_i \cos\beta_i$ and $v_i = k_i \sin\beta_i$. Subscripts are used to denote the frequency components, and superscripts for the perturbation orders.

The total potential, correct to second order, is $\phi = \phi^{(1)} + \phi^{(2)}$. The first- and second-order potentials are given by the real parts of

$$\phi^{(1)} = A_1\phi_1 \mathrm{e}^{\mathrm{i}\omega_1 t} + A_2\phi_2 \mathrm{e}^{\mathrm{i}\omega_2 t}\,, \tag{1}$$

$$\phi^{(2)} = A_1^2\phi_{11}^+ \mathrm{e}^{2\mathrm{i}\omega_1 t} + A_2^2\phi_{22}^+ \mathrm{e}^{2\mathrm{i}\omega_2 t} + 2A_1A_2\phi_{12}^+ \mathrm{e}^{\mathrm{i}(\omega_1+\omega_2)t} + 2A_1A_2^*\phi_{12}^- \mathrm{e}^{\mathrm{i}(\omega_1-\omega_2)t}\,. \tag{2}$$

The incident waves propagate toward the body in the domain $x < 0$, thus $|\beta_i| < \pi/2$ and $u_i > 0$. Since the wavelength is short relative to the dimensions of the body, complete reflection is assumed for the first-order solution at the waterline $x = -b$. Thus, in the domain $x \le -b$,

$$\phi_i \simeq \frac{2g}{\omega_i}\cos\left(u_i(x+b)\right)\mathrm{e}^{k_i z + \mathrm{i}u_i b - \mathrm{i}v_i y}\,. \tag{3}$$

The first-order solution vanishes to leading order for $x \ge b$ and in the region below the body.

3 The second-order free-surface condition

The potential $\phi^{(2)}$ satisfies the homogeneous condition $\phi_n^{(2)} = 0$ on the submerged surface of the body, and the inhomogeneous free-surface condition

$$\phi_{tt}^{(2)} + g\phi_z^{(2)} = -\frac{\partial}{\partial t}\left(\nabla\phi^{(1)}\right)^2 \quad \text{on} \quad z = 0 \quad \text{and} \quad |x| > b\,. \tag{4}$$

The right-hand side of (4) is simplified since the first-order potentials are of the form (3).

Since the second-harmonic terms involving $\mathrm{e}^{2\mathrm{i}\omega_i t}$ can be recovered from the sum-frequency term as special cases, we consider only the last two terms in (2). After substituting (3) in (4) and performing some reduction the second-order free-surface condition takes the form

$$\begin{aligned} -(\omega_1 \pm \omega_2)^2\phi_{12}^{(\pm)} + g\phi_{12z}^{(\pm)} = -\mathrm{i}H(-x)(\omega_1 \pm \omega_2)\frac{g^2}{\omega_1\omega_2}\mathrm{e}^{-\mathrm{i}(v_1 \pm v_2)y + \mathrm{i}(u_1 \pm u_2)b} \\ \Big[(k_1k_2 \mp v_1v_2 - u_1u_2)\cos((u_1+u_2)(x+b)) \\ +(k_1k_2 \mp v_1v_2 + u_1u_2)\cos((u_1-u_2)(x+b))\Big]\,. \end{aligned} \tag{5}$$

Here $H(-x)$ is the Heaviside unit function, which vanishes for $x > 0$. In general there are two components of the forcing function on the right side of (5) which are oscillatory in the x-direction with the wavenumbers $u_1 \pm u_2$. These two wavenumbers are present for both the sum- and difference-frequency cases. In special cases, where the factor $(k_1k_2 \mp v_1v_2 - u_1u_2) = 0$, only one component exists with the 'slow' wavenumber $u_1 - u_2$. This follows in all cases of normal incidence, where $v_i = 0$ and $u_i = k_i$, and also for the sum-frequency case in oblique monochromatic waves. The slow component is particularly important when $u_1 - u_2 \to 0$ since the forcing is nearly constant, extending to infinity, and this results in a second-order solution which persists to large depths in the fluid.

4 Particular solutions of the free-surface condition

Solutions of the free-surface condition (5) can be decomposed into components which are solutions of

$$-\nu\varphi + \varphi_z = H(-x)\mathrm{e}^{-\mathrm{i}ux-\mathrm{i}vy} \quad \text{on} \quad z = 0\,. \tag{6}$$

Particular solutions which satisfy (6) can be combined so that the remaining components of the second-order potential satisfy either homogeneous boundary conditions on the free surface or inhomogeneous conditions where the forcing functions on the right-hand-side tend to zero away from the body.

Solutions of (6) can be constructed from the potential for a pressure distribution on the free surface (Wehausen & Laitone, 1960, equation 21.3). The forcing function on the right side of (6) is first restricted to a finite rectangular domain $(-M < x < 0,\ -M < y < +M)$, and the limit as $M \to \infty$ is evaluated. This gives the solution in the form

$$\varphi = \frac{\mathrm{i}}{2\pi}\mathrm{e}^{-\mathrm{i}vy}\int_C \frac{\mathrm{e}^{kz+\mathrm{i}rx}\,dr}{(k-\nu)(r+u)}\,, \tag{7}$$

where $k = \sqrt{r^2+v^2}$ and C is an appropriate contour of integration between $\mp\infty$. The pole at $r = -u$ is associated with the 'locked waves', which propagate with the same phase as the forcing function in (6). Defining the contour C to pass above this pole ensures that the integration from $-M$ to 0 tends to a finite limit as $M \to \infty$. Two other poles are associated with the 'free waves', where $k = \nu$ and $r = \pm\sqrt{\nu^2 - v^2} \equiv \pm\mu$. The radiation condition requires that $\mathrm{Im}(k) > 0$, and thus when μ is real the contour C passes above the pole $r = +\mu$ and below the pole $r = -\mu$. Except for these three poles, and branch points at $r = \pm\mathrm{i}v$ associated with the function $k = \sqrt{r^2+v^2}$, the integrand of (7) is analytic in the complex r-plane. Branch cuts are established extending from $\pm\mathrm{i}|v|$ to $\pm\mathrm{i}\infty$ on the imaginary axes, and k is continued into the cut plane with the convention that $k > 0$ on the real axis.

For $x \gtrless 0$ the contour C in (7) can be replaced by a contour around the upper or lower branch cut, respectively. It follows from residue theory that

$$\varphi = \left[\pm\frac{1}{2\pi}I(\pm u) + \frac{\nu(u\pm\mu)}{\mu(\kappa^2-\nu^2)}\mathrm{e}^{\nu z-\mathrm{i}\mu|x|} + H(-x)\frac{\mathrm{e}^{\kappa z-\mathrm{i}ux}}{\kappa-\nu}\right]\mathrm{e}^{-\mathrm{i}vy}\,, \tag{8}$$

where the sign $(\pm)$ corresponds to the domain $x \gtrless 0$ and

$$I(u) = \int_{|v|}^{\infty} \frac{\mathrm{e}^{-t|x|}}{t-\mathrm{i}u}\left[\frac{\mathrm{e}^{\mathrm{i}wz}}{w+\mathrm{i}\nu} + \frac{\mathrm{e}^{-\mathrm{i}wz}}{w-\mathrm{i}\nu}\right]dt\,. \tag{9}$$

Here $\kappa = \sqrt{u^2+v^2}$ and $w = \sqrt{t^2-v^2}$. When $u = 0$ and $v = 0$, corresponding to the limit where the oscillatory part of the forcing function on the right side of (6) is constant, (9) reduces to the integral representation derived in a different manner by Miao & Liu (1989, equation 14).

For large values of $|\nu x|$ and $|\kappa x|$ I tends to zero, exponentially for $|v| > 0$ and in proportion to $|x|^{-2}$ when $v = 0$. In these cases the first term in (8) is evanescent. However when $\kappa = \sqrt{u^2+v^2} = 0$, the combination of the first and third terms is vortex-like in the far field, as shown in the next Section. This is the dominant cause of the second-order force in short wavelengths. The second term in (8) represents radiating free waves on both sides of $x = 0$. The locked waves represented by the third term exist only in $x < 0$.

For the sum-frequency case, where $\nu = (\omega_1+\omega_2)^2/g$, it is easy to show that $\nu^2 > (u^2+v^2)$; thus $(\mu^2-u^2) = (\nu^2-\kappa^2) > 0$, and μ is real. For the difference-frequency case, $\nu = (\omega_1-\omega_2)^2/g$, regimes exist where $\nu^2 < v^2$ and μ is imaginary. In these regimes the poles at $r = \pm\mu$ are on the imaginary axis, and the first exponential in (8) is replaced by $\mathrm{e}^{\nu z-|\mu x|}$; thus the free waves are trapped, with exponential attenuation in both the $\pm x$-directions.

5 Normal incidence

For normal incidence, where $v_i = 0$ and $u_i = k_i$,

$$I(u) = \frac{\mathrm{i}}{\nu^2 - u^2}\Big\{(\nu - u)\left[\mathrm{e}^{\mathrm{i}\nu\zeta^*}E_1(\mathrm{i}\nu\zeta^*) - \mathrm{e}^{-\mathrm{i}u\zeta^*}E_1(-\mathrm{i}u\zeta^*)\right]$$
$$-(\nu + u)\left[\mathrm{e}^{-\mathrm{i}\nu\zeta}E_1(-\mathrm{i}\nu\zeta) - \mathrm{e}^{-\mathrm{i}u\zeta}E_1(-\mathrm{i}u\zeta)\right]\Big\}. \tag{10}$$

Here E_1 denotes the exponential integral and $\zeta = |x| + \mathrm{i}z$. For monochromatic waves $|u| = |u_1 - u_2| \to 0$, and the limiting value of (8) is

$$\varphi(0) = -\frac{\pi - 2\theta}{2\pi\nu} + \frac{\mathrm{sgn}(x)}{2\pi\mathrm{i}\nu}\left\{\mathrm{e}^{-\mathrm{i}\nu\zeta}E_1(-\mathrm{i}\nu\zeta) - \mathrm{e}^{\mathrm{i}\nu\zeta^*}E_1(\mathrm{i}\nu\zeta^*)\right\} - \frac{\mathrm{sgn}(x)}{\nu}\mathrm{e}^{\nu z - \mathrm{i}\nu|x|}, \tag{11}$$

where $\theta = \tan^{-1}(x/|z|)$.

An interesting connection exists between the particular solution (11) and the 'line vortex potential' λ derived by McIver (1994, equation 24). Both are harmonic functions which satisfy the same free-surface condition, and thus they differ by a homogeneous solution of the free-surface condition. Using relations given by Wehausen & Laitone (1960, equations 13.28-31), it can be shown that (11) is equivalent to the potential of a point vortex at $x = 0$, $z = 0$, and the difference between (11) and McIver's λ is a horizontal dipole at the same point.

6 Applications

Results will be shown comparing the second-order forces obtained from these approximate solutions with computations for three-dimensional bodies carried out using the second-order extension of the panel code WAMIT. In some cases the agreement is sufficiently good to provide a useful quantitative estimate. In other cases the practical value of the approximation is only qualitative.

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