WAVE IMPACT ON CRACKED ELASTIC BEAM

T.I. Khabakhpasheva

Lavrentyev Institute of Hydrodynamics, Novosibirsk, 630090, RUSSIA

ABSTRACT: The impact of an elastic cracked plate of finite length that is dropped against a liquid free surface is analyzed. The problem is considered within the Wagner theory. The liquid flow is two-dimentional, symmetric and potential. The presence of a crack is modelled with the help of a torsional linear spring. The analysis is based on the normal mode method with hydroelastic behaviour of the plate being of the main interest. The numerical results demonstrate that the presence of a crack has important effects on bending stresses and deflections, both at the impact stage and at the penetration stage, and may significantly change the life time of a ship structure.

1 INTRODUCTION

During the process exploitation floating structures may suffer from severe wave impact. In spite of the fact that the water impact is an event of a short duration, it is accompanied by high hydrodynamic loads which may damage full construction, especially in the case of periodic wave impact loads. Moreover, in reality the structures contain initially presented small flaws (cracks, cavities and inclusions) which can absorb a part of the impact energy during fluid-structure interaction and grow with time. This analysis is mainly motivated by ship hydrodynamics, where wave impact onto the wetdeck of a catamaran can be very severe and may lead to local structural damage.

Here the plane unsteady problem of wave impact upon a cracked elastic beam of finite length is considered. Specifically, we consider a wave crest that touches the beam at its central point, where a crack is located, and assume that the liquid free surface hits the beam from below with a constant velocity. We determine both the deflection of the beam and the stress distribution in the beam, and investigate the effects the length of the crack and other impact parameters (notably, radius of curvature of the wave and impact velocity).

The method of matched asymptotic expansions is used to account for the effect of the crack. According to this method, the beam is divided into an inner region that surrounds the crack, and an outer region where transverse variations of the stresses are not important and the plate is modelled as an elementary homogeneous beam. The presence of a crack is modelled with the help of a linear spring.

Analysis of the impact process is based on hydroelasticity, in which the coupled hydrodynamics and structural dynamics problems are solved simultaneously. Even after all possible simplifications at the impact stage the problem remains nonlinear, because the dimension of the contact region is unknown in advance and has to be determined together with the liquid flow and the beam deflection.

The problem is analyzed using the normal mode method. With appropriate modifications, this method leads to an infinite system of ordinary differential equations with respect to principal coordinates of the beam deflection and dimensions of the contact region. In this work, the cracked-beam elastic reaction is of main interest, and the numerical method is accordingly designed to permit effective evaluation of the elastic (rather than the hydrodynamic) characteristics.

The numerical results demonstrate that the presence of a crack has important effects on bending stresses and deflections, both at the impact stage and at the penetration stage, and may significantly change the life time of a ship structure.

2 STATEMENT OF THE PROBLEM

The plane unsteady problem of wave impact upon a cracked elastic beam of finite length is considered. Liquid is supposed to be ideal and incompressible. Initially ($t' = 0$) a wave crest touches the beam at its central point. Then the liquid starts to move up with a constant velocity V. The initial contact point is taken as the origin of the Cartesian coordinate system $x'Oy'$ (dimensional variables are denoted by a prime). The curve $y' = -x'^2/2R$ corresponds to the liquid free surface at $t' = 0$. This curve describes the shape of the wave crest with radius of the curvature R. The plate is supposed to be cracked at the central point, so the flow caused by the plate impact is symmetric with respect to the line $x' = 0$ (see Figure 1).

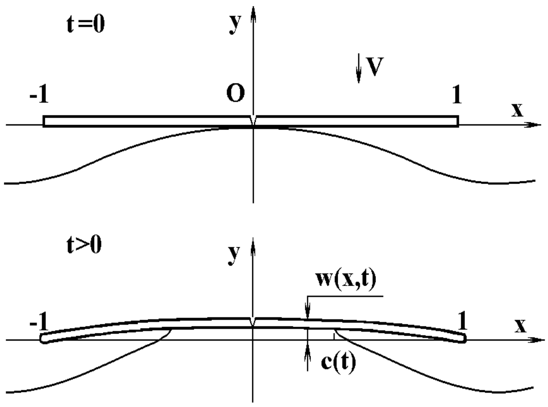


Figure 1.

In order to model the cracked beam behaviour, the method of matched asymptotic expansions is used. According to this method, the beam is divided into the 'inner' region which surrounds the crack, and the 'outer' region, where the transverse variation of the stresses is not important and the plate is modeled by homogeneous beam. In the framework of the linear theory the presence of a crack is modeled with the help of a torsional spring (see Figure 2). The stiffness K_T of the equivalent torsional spring for a single-sided crack is assumed known as a function of the beam parameters and the crack length a. Non-dimensional stiffness $k_T = K_T L/EJ$ is presented in Rizos *at all* (1990) and depicted in Figure 3. The solution for the beam, which is divided by the torsional spring into two parts, provides the bending stresses outside the crack region. This solution together with the results by Bueckner (1960) makes it possible to evaluate the stress distribution near the crack. This problem is not considered here.

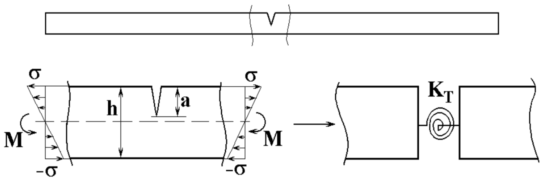


Figure 2.

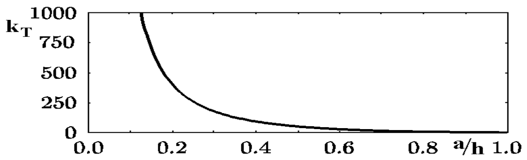


Figure 3.

3 MATHEMATICAL FORMULATION

Non-dimensional variables are used below. The beam length L is taken as the length scale and the impact velocity V as the velocity scale of liquid particles. $L^2/(RV)$ is taken as the time scale, L^2/R as the displacement scale, $\rho V^2(R/L)$ as the pressure scale, where ρ is the liquid density.

The original equation of liquid flow, the boundary and initial conditions and the Euler beam equation, which are written in the non-dimensional variables, contain three parameters $\alpha = M_B/(\rho L)$, $\beta = (EJ)/(\rho L R^2 V^2)$ and $k_T = K_T L/EJ$. Here M_B is the beam mass per unit length, E is the elasticity modulus, $J = h^3/12$ is the inertia momentum of the beam cross-section, h is a thickness of the beam, k_T is the dimensionless local flexibility coefficient, K_T is spring stiffness.

The plane and potential flow generated by the plate penetration and the plate behaviour are described by the velocity potential $\varphi(x, y, t)$ and the beam deflection $w(x, t)$ which satisfy the following equations

$$\varphi_{xx} + \varphi_{yy} = 0 \qquad (y < 0), \tag{1}$$

$$\varphi_y = -1 + w_t(x, t) \qquad (y = 0, \quad |x| < c(t)), \tag{2}$$

$$\varphi = 0 \qquad (y = 0, \quad |x| > c(t)), \tag{3}$$

$$\varphi \to 0 \qquad (x^2 + y^2 \to \infty), \tag{4}$$

$$p(x, y, t) = -\varphi_t(x, y, t), \tag{5}$$

$$\alpha \frac{\partial^2 w}{\partial t^2} + \beta \frac{\partial^4 w}{\partial x^4} = p(x, 0, t) \qquad (|x| < 1, \quad t > 0), \tag{6}$$

$$w = 0, \qquad w_{xx} = 0 \qquad (x = \pm 1, \quad t > 0), \tag{7}$$

$$w = w_t = 0 \qquad (|x| < 1, \quad t = 0). \tag{8}$$

The presence of the crack is described by conditions

$$w(-0) = w(+0), \quad w_{xx}(-0) = w_{xx}(+0), \quad w_{xxx} = 0,$$
$$w_{xx} - 2k_T w_x = 0 \qquad (x = 0, \quad t > 0). \tag{9}$$

The bending stress distribution $\sigma(x, t)$ is given in the dimensionless variables as $\sigma(x, t) = w_{xx}(x, t)$, with its scale $Eh/(2R)$. The positions of the contact points (liquid-beam-air) are described in the symmetrical case by the only function $c(t)$. Despite the fact that both the equations of motion and the boundary conditions are linearized, the problem remains nonlinear as $c(t)$ is unknown.

The formulation of the problem (1) - (9) is not complete. It must be added by an equation for the dimension of the contact region. Usually the equation derived by Wagner (1932) is used, but this equation is difficult to incorporate into a numerical scheme. Here the equation suggested by Korobkin (1996) is used. The equation is, in fact, a modification of the classical Wagner condition. It has the form

$$\int_0^{\pi/2} y_b[c(t)\sin\theta, t]d\theta = 0, \tag{10}$$

where the function $y_b(x, t)$ describes the shape of the beam with respect to the initial position of the free surface. In the present case, $y_b(x, t) = x^2/2 - t + w(x, t)$, and equation (10) gives

$$t = \frac{1}{4}c^2 + \frac{2}{\pi}\int_0^{\pi/2} w[c(t)\sin\theta, t]d\theta. \tag{11}$$

The hydrodynamic part (1) - (5), the structural part (6) - (9) and the geometrical part (10) of the Wagner problem are closely connected to each other and have to be treated simultaneously in general case.

Wagner problem (1) - (10) is solved with the help of the normal mode method.

4 NORMAL MODE METHOD

Within this method the beam deflection $w(x, t)$ is sought in the form

$$w(x, t) = \sum_{n=1}^{\infty} a_n(t)\psi_n(x), \tag{12}$$

where eigenfunctions $\psi_n(x)$ represent the eigen modes of simply supported beam - so called 'dry' modes. In the case of the cracked beam they are given as

$$\psi_n(x) = A_n[\sin\lambda_n(|x| - 1) + \sinh\lambda_n(|x| - 1)\frac{\cos\lambda_n}{\cosh\lambda_n}],$$

$$A_n = \left(1 - \frac{\cos^2\lambda_n}{\cosh^2\lambda_n} + \frac{\sin 2\lambda_n}{2\lambda_n} - \frac{\tanh\lambda_n\cos^2\lambda_n}{\lambda_n}\right)^{-1/2}$$

λ_n are the solutions of the equation $4k_T = \lambda_n(\tanh\lambda_n - \tan\lambda_n)$. It is convenient to take the principal coordinates $a_n(t)$ of the beam deflection as the new unknown functions and to express other quantities with their help Khabakhpasheva & Korobkin (1997).

As a result, we arrive at the infinite system of ordinary differential equations with respect to the principal coordinates $\vec{a} = (a_1, a_2, a_3, ...)^T$ and auxiliary vector-function $\vec{v} = (v_1, v_2, v_3, ...)^T$:

$$\frac{d\vec{a}}{dt} = (\alpha I + S)^{-1}(\beta D\vec{v} + \vec{f}), \tag{13}$$

$$\frac{d\vec{v}}{dt} = -\vec{a}, \tag{14}$$

where I is the unit matrix and D is the diagonal matrix, $D = \mathrm{diag}\{\lambda_1^4, \lambda_2^4, \lambda_3^4, ...\}$,

$$f_m(c) = \int_{-c}^{c} \sqrt{c^2 - x^2}\psi_m(x)\, dx,$$

$$S_{nm}(c) = \int_{-c}^{c} \varphi_n(x, 0, c)\psi_m(x)\, dx.$$

Here $\varphi_n(x, y, c)$ is a harmonic in the lower half-plane function, which satisfies equations (1)-(4) with the right part of equation (2) being replaced by the function $\psi_n(x)$. Integrals $f_n(c)$ have the form:

$$f_n(c) = \frac{A_n \pi c \cos\lambda_n}{\lambda_n} \left[\mathbf{H_1}(\lambda_n c) - J_1(\lambda_n c)\tan\lambda_n + \right.$$
$$\left. + \mathbf{L_1}(\lambda_n c) - I_1(\lambda_n c)\tanh\lambda_n\right].$$

Here and below $\mathbf{H_0}(z)$ and $\mathbf{H_1}(z)$ are the zero- and first-order Struve functions, $\mathbf{L_0}(z)$ and $\mathbf{L_1}(z)$ are zero- and first-order modify Struve function, $J_0(z)$ and $J_1(z)$ are the zero- and first-order Bessel functions and $I_0(z)$ and $I_1(z)$ are the zero- and first-order modify Bessel function.

The integrals $S_{nm}(c)$ were evaluated earlier in the case $\psi_n(x) = \cos(\mu_n x)$, $\mu_n = \pi(n - \frac{1}{2})$, which corresponds to the problem of wave impact onto the center of a homogeneous elastic plate (Korobkin 1998). The corresponding functions which are denoted by $\tilde{S}_{nm}(c)$ below, are given as

$$\tilde{S}_{nm}(c) = \frac{\pi c}{\mu_n^2 - \mu_m^2} \left[\mu_n J_0(\mu_m c) J_1(\mu_n c) - \right.$$
$$\left. - \mu_m J_0(\mu_n c) J_1(\mu_m c)\right] \quad (n \neq m),$$
$$\tilde{S}_{nn}(c) = \frac{\pi}{2} c^2 \left[J_0^2(\mu_n c) + J_1^2(\mu_n c)\right].$$

In order to evaluate the integrals $S_{nm}(c)$ in the problem under consideration, we use the expansion

$$\psi_m(x) = \sum_{n=1}^{\infty} C_{nm} \cos(\mu_n x), \qquad (|x| < 1, n \geq 1).$$

It is worth to notice that the matrix C with the elements C_{nm}, where $n, m = 1, 2, ..$, is orthogonal, which means that $\det(C) = 1$ and $C^{-1} = C^*$. We find

$$C_{nm}(c) = A_m \left[\frac{2\lambda_m \cos\lambda_m}{\lambda_m^2 - \mu_n^2} - \frac{2\lambda_m \cos\lambda_m}{\lambda_m^2 + \mu_n^2}\right].$$

The right-hand side of system (13) and (14) depends on $\vec{a}$, $\vec{v}$ and c but not on the time t, which is why it is convenient to take the quantity c as a new independent variable ($0 \leq c \leq 1$) instead of time t. Time t is the function of c now, $t = t(c)$. This substitution is justified under the condition $dc/dt > 0$, which is the main assumption within the Wagner approach. Differential equation for the unknown function $t(c)$ follows from equation (11) after its differentiation with respect to c and taking into account expansion (12)

$$\frac{dt}{dc} = Q(c, \vec{a}, \dot{\vec{a}}), \tag{15}$$

$$Q(c, \vec{a}, \dot{\vec{a}}) = \frac{c/2 + 2/\pi \int_0^{\pi/2} w_x[c\sin\theta, t]\sin\theta d\theta}{\pi/2 - \int_0^{\pi/2} w_t[c\sin\theta, t] d\theta} =$$
$$= \frac{2/\pi + (\vec{a}, \vec{\Gamma}_1)}{1 - (\dot{\vec{a}}, \vec{\Gamma}_0)},$$

$$\Gamma_{0n} = A_n \cos(\lambda_n) \left[\mathbf{H_0}(\lambda_n c) - J_0(\lambda_n c)\tan\lambda_n + \right.$$
$$\left. + \mathbf{L_0}(\lambda_n c) - I_0(\lambda_n c)\tanh\lambda_n\right],$$

$$\Gamma_{1n} = A_n \lambda_n \cos(\lambda_n) \left[\frac{4}{\pi} + J_1(\lambda_n c)\tan\lambda_n - \right.$$
$$\left. - \mathbf{H_1}(\lambda_n c) + \mathbf{L_1}(\lambda_n c) - I_1(\lambda_n c)\tanh\lambda_n\right].$$

Multiplying equations of system (13), (14) by dt/dc and taking (15) into account, we find

$$\frac{d\vec{a}}{dc} = \vec{F}(c, \vec{v})\, Q(c, \vec{a}, \vec{F}(c, \vec{v})), \tag{16}$$
$$\frac{d\vec{v}}{dc} = -\vec{a}\, Q(c, \vec{a}, \vec{F}(c, \vec{v})),$$

where $\vec{F}(c, \vec{v}) = (\alpha I + S(c))^{-1}(\beta D\vec{v} + \vec{f}(c))$ and $\dot{\vec{a}} = \vec{F}(c, \vec{v})$.

The initial conditions for the system of ordinary differential equations (15)-(16) are

$$\vec{a} = 0, \quad \vec{v} = 0, \quad t = 0 \quad (c = 0). \tag{17}$$

The initial-value problem (15)-(17) is suitable for numerical simulations of hydroelastic behavior of the wave impact on the cracked plate.

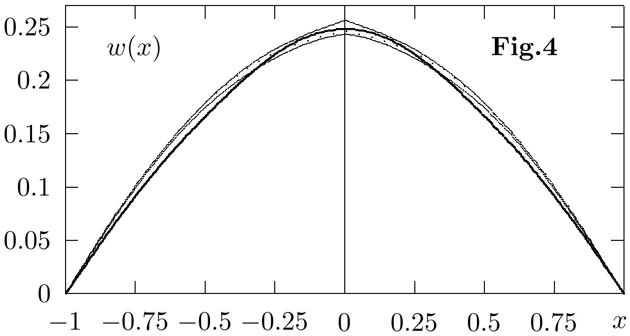
5 NUMERICAL RESULTS AND DISCUSSION

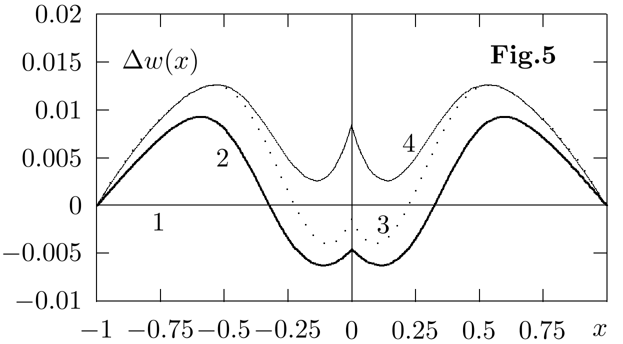
The initial-value problem (15)-(17) is solved numerically by the fourth-order Runge-Kutta method with uniform step Δc. The condition that the numerical scheme is stable was derived. The step Δc has to decrease as $O(N^{-2})$ if the number of modes N taken into account increases. Calculations were performed for the case $L = 0.5$m, $R = 10$m, $h = 2$cm, $E = 21 \cdot 10^{10}$H/m^2, $V = 3$m/s, $\varrho = 1000$kg/m^3, $\varrho_b = 7850$kg/m^3 where ϱ_b is the beam density. This gives $\alpha = 0.314$, $\beta = 0.311$. The number of 'dry' modes N taken into account is equal to 5. Numerical results for homogeneous plate are compared with those for cracked plate ($a/h = 0.5$ and $a/h = 0.8$) and for almost broken plate ($a/h = 0.99$).

Analysis of numerical results obtained for different lengths of the crack gives:

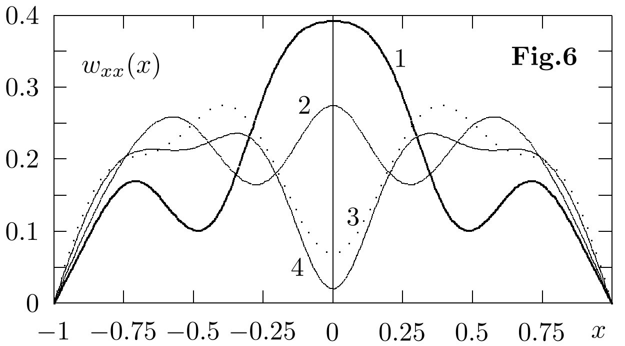
- The evolution in time of both the contact point $c(t)$ and its velocity dc/dt are weakly dependent of the crack length and are almost the same as for a homogeneous beam without a crack (see Korobkin (1998)).
- At the end of the impact stage, when the plate is totally wetted, the maximum deflection occurs at the plate midpoint. The maximum deflection is not a monotonic function of the crack length (Fig. 4, 5). However for the almost broken plate ($a/h = 0.99$), the maximum deflection is greater than for the corresponding homogeneous plate ($a = 0$). It is seen that for the cracked plate the deflection at one quarter of the plate length is greater than the deflection of the homogeneous plate at the same points.

The difference between the curves in Fig. 4 is shown in Fig. 5 with the deflection of the homogeneous plate taken as the reference. Here and below line 1 is for $a/h = 0$, line 2 is for $a/h = 0.5$, line 3 is for $a/h = 0.8$ and line 4 is for $a/h = 0.99$.



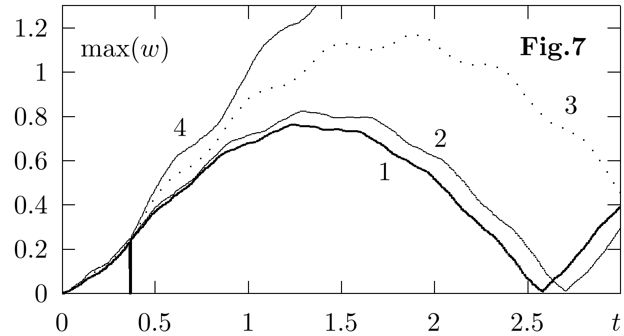


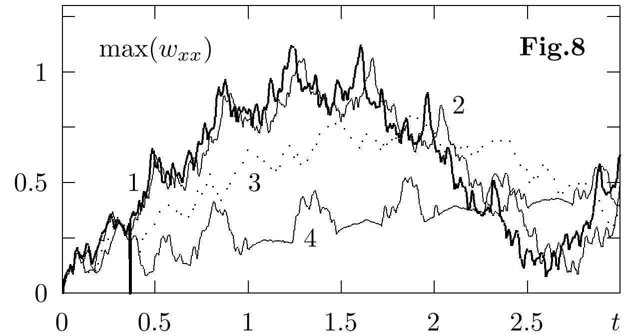
• Fig. 6 shows the distributions of the bending stresses along the plate for different crack lengths. It is seen that the distributions are strongly dependent of the crack length. Note that the bending stress distribution for the homogeneous plate has three local maxima with



the greatest at the plate midpoint. Two other maxima are close to the plate edges and their amplitudes are about three times smaller than the stress amplitude at the plate center. With the crack length increase, the amplitudes of the "edge" maxima grow and the locations of the maxima shift to the plate center. At the same time the bending stresses at the plate midpoint monotonically decrease down to zero at $a = h$.

• At the penetration stage the crack presence becomes even more important. With the crack length increase, both the period and the amplitude of hydroelastic vibrations of the plate increase (Fig. 7), however, the amplitude of the bending stresses decreases (Fig. 8). Figure 8 demonstrates also that the longer the crack, the greater the contribution of the higher modes to the stresses at the penetration stage.





CONCLUSION

It was shown that presence of a crack is important for evaluation of bending stresses and deflections in the impacting plate. Repeated hydrodynamic loads during the structure-wave interaction may lead not only to growing of the crack but also to the damage of the whole construction. If we know the bending moment distribution along the beam and use the fatigue damage theory, we can estimate number of water impacts which construction may stands, i.e. estimate lifetime of a ship structure.

Acknowledgement

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