Free and forced oscillations to second order for two-dimensional fluid motion in a tank with arbitrary bed profile

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Introduction

In this paper we consider solutions at first and second order in the Stokes' expansion of the velocity potential for two-dimensional free and forced oscillations of a fluid in a vertically-walled tank having an arbitrary bedform.

Free oscillations, or sloshing, of a fluid in a tank at first order is a classical eigenvalue problem of fluid mechanics which, despite its long history and the illustrious names involved with it, remains an active area of research. This is driven by the need to understand when resonance is likely to occur in partially-filled containers such as fuel tankers. Fox & Kuttler (1983) provide an extensive review of the two-dimensional sloshing problem and cite many important references on this subject. Under small amplitude external forcing, the first order, or linear, sloshing frequencies provide a good estimate of when resonance will occur. For larger amplitude forcing, or when the forcing is close to resonance, non-linear effects undoubtedly play a significant role and there is an important area of research based on the non-linear and often violent motion that occurs. See, for example, Faltinsen *et al* (2000) who use a multi-dimensional modal approach on a generalised domain and surface modes rather than natural modes. The sloshing problem is also amenable to fully non-linear solvers, see for example Wu & Eatock Taylor (1994) who apply their finite-element method code to consider the sloshing problem in a rectangular tank.

In much of the work on sloshing of a fluid in a tank, the base of the tank is taken to be flat. Here, we consider the case where the bed of the tank can be of arbitrary profile and develop a technique for the first order problem based on that used by Porter & Porter (2000). Thus, the Green's function for a flat bed is used in Green's identity in conjunction with the Cauchy-Riemann equations to derive integral equations for functions relating to the tangential flux along the varying bed. Furthermore, we extend the sloshing problem to second-order, developing a solution technique based in part on the technique used at first order. Although much more complicated, it is shown that the formulations at both first and second-order are exact although the integral equations that arise must inevitably be solved numerically and this is achieved using an efficient and accurate Galerkin method. Numerical results confirm established results for sloshing frequencies at first-order for specific geometries such as triangular beds and semi-circular troughs.

Formulation of the sloshing problem

Cartesian co-ordinates x, y are chosen with y directed vertically downwards and $y = 0$ coinciding with the undisturbed free surface of the fluid. The tank has vertical walls at $x = 0, l$ and the bed, C, is defined by $y = h(x)$ with the constant depth d defined as $d = \max\{h(x) | x \in [0, l]\}$. We start from the usual form of the Stokes' expansion of the velocity potential $\Phi(x, y, t)$ up to second order, writing $\Phi = \epsilon\Phi_1 + \epsilon^2\Phi_2$ where ϵ is a small parameter and assume a time harmonic variation in the first-order potential of frequency ω, so that

$$\Phi_1(x, y, t) = \mathrm{Re}\left\{\phi_1(x, y; K)\mathrm{e}^{-\mathrm{i}\omega t}\right\} \tag{1}$$

where the frequency parameter is $K = \omega^2/g$. Hence ϕ_1 satisfies

$$\nabla^2\phi_1 = 0, \qquad \text{in the fluid domain, } D \tag{2}$$

$$\frac{\partial\phi_1}{\partial x} = 0, \qquad \{x = 0, 0 < y < h(0)\} \cup \{x = l, 0 < y < h(l)\} \tag{3}$$

$$K\phi_1 + \frac{\partial\phi_1}{\partial y} = 0, \qquad y = 0,\ 0 < x < l \tag{4}$$

$$\frac{\partial\phi_1}{\partial n} = 0, \qquad y = h(x),\ 0 < x < l. \tag{5}$$

where $\partial/\partial n = \mathbf{n}.\nabla$ and $\mathbf{n}$ is the outward normal on C. The form of the second-order free surface boundary condition (FSBC) suggests that the second-order potential has the form

$$\Phi_2(x, y, t) = \Phi_s(x, y) - \Gamma t + \mathrm{Re}\{\phi_2(x, y)\mathrm{e}^{-2\mathrm{i}\omega t}\} \tag{6}$$

where the steady and double frequency components of the potential, Φ_s and ϕ_2, both satisfy Laplace's equation, (2) and the zero-flux conditions (3) and (5) on the fixed boundaries. The choice of Γ simply affects the position of the mean free-surface and is set to a value depending upon ϕ_1 which guarantees mass conservation at second-order. Furthermore, we are able to deduce from the second order FSBC that $\Phi_s = 0$ and that

$$4K\phi_2 + \frac{\partial\phi_2}{\partial y} = g(x) \equiv -\frac{\mathrm{i}\omega}{g}\left\{\left(\frac{\partial\phi_1}{\partial x}\right)^2 + \frac{3}{2}K^2\phi_1^2 + \frac{1}{2}\phi_1\frac{\partial^2\phi_1}{\partial x^2}\right\}_{y=0} \qquad y = 0,\ 0 < x < l. \tag{7}$$

In the solution to both the first-order and second-order problems we will make use of a Green's function, $G(x,y|x_0,y_0;K)$, which satisfies

$$\nabla^2 G = -\delta(x-x_0)\delta(y-y_0) \tag{8}$$

$$\frac{\partial G}{\partial x} = 0, \qquad x = 0, l,\ 0 < y < d \tag{9}$$

$$KG + \frac{\partial G}{\partial y} = 0, \quad \text{on } y = 0, \qquad \text{and} \qquad \frac{\partial G}{\partial y} = 0, \quad \text{on } y = d, \tag{10}$$

for a rectangular tank with a flat bed and a homogeneous free surface condition. It can be shown that

$$G(x,y|x_0,y_0;K) = \sum_{n=0}^{\infty} \frac{\psi_n(y)\psi_n(y_0)}{2k_n d} \frac{\{\cosh k_n(l-|x-x_0|) + \cosh k_n(l-x-x_0)\}}{\sinh k_n l}. \tag{11}$$

where, for $n = 0, 1, 2, \ldots$,

$$\psi_n(y) = N_n^{-1/2} \cos k_n(d-y) \qquad \text{and} \qquad N_n = \tfrac{1}{2}\{1 + \sin(2k_n d)/2k_n d\} \tag{12}$$

and we have used k_n $(n = 1, 2, \ldots)$ to denote the positive real roots of $K = -k_n \tan k_n d$ incorporating $k_0 = -\mathrm{i}k$ where k is the real root of the dispersion relation $K = k \tanh kd$.

First and second order solutions

Applying Green's Identity to $\phi_1(x,y;K)$ and $G(x,y|x_0,y_0;K)$ in the fluid domain results in

$$\phi_1(x_0,y_0) = -\int_C \phi_1 \frac{\partial G}{\partial n}\, \mathrm{d}s. \tag{13}$$

where s measures the arc length on C , the curve $y = h(x)$. This integral equation for ϕ_1 can be used to determine the sloshing frequencies, but we proceed further following Porter & Porter's (2000) technique of converting normal derivatives to tangential derivatives by using the following relations

$$\frac{\partial}{\partial s}\psi_n(y)\mathrm{e}^{\pm k_n x} = \mp\frac{\partial}{\partial n}\chi_n(y)\mathrm{e}^{\pm k_n x} \qquad \text{and} \qquad \frac{\partial}{\partial n}\psi_n(y)\mathrm{e}^{\pm k_n x} = \pm\frac{\partial}{\partial s}\chi_n(y)\mathrm{e}^{\pm k_n x}, \tag{14}$$

where $\partial/\partial s = \mathbf{s}.\nabla$ and $\mathbf{s}$ is the unit vector tangential to the curve C and $\chi_n(y) = N_n^{-1/2} \sin k_n(d-y)$. The relations (14) are in effect Cauchy-Riemann relations and are thus restricted in their application to two-dimensional problems. Using these relations we deduce that

$$\frac{\partial^2 G}{\partial n \partial n_0} = -\frac{\partial^2 H}{\partial s \partial s_0}$$

where

$$H(x,y|x_0,y_0;K) = \sum_{n=0}^{\infty} \frac{\chi_n(y)\chi_n(y_0)}{2k_n d} \frac{\{\cosh k_n(l-|x-x_0|) - \cosh k_n(l-x-x_0)\}}{\sinh k_n l}. \tag{15}$$

We can now derive an alternative integral equation by differentiating (13) with respect to n_0 and noting that this derivative vanishes on $y_0 = h(x_0)$. We convert normal to tangential derivatives and after performing integration by parts, noting that $\phi_1 \to 0$ as $(x,y) \to (0,h(0)),(l,h(l))$ to eliminate free terms and eventually obtain

$$\int_C H \frac{\partial \phi_1}{\partial s}\, \mathrm{d}s = 0. \tag{16}$$

After changing variables from s to x , we have

$$\int_0^l m(x_0,x;K) q_1(x)\, \mathrm{d}x = 0, \qquad 0 < x_0 < l \tag{17}$$

where $m(x_0,x;K) = H(x,h(x)|x_0,h(x_0);K)$ and $q_1(x)$ is a function which is proportional to the tangential flux of the fluid along the curve C . Non-trivial solutions of this homogeneous first kind integral equation furnish the sloshing frequencies, ω for the tank containing the particular bed shape $y = h(x)$ and the corresponding eigenfunction $q_1(x)$.

To make progress at second order we note from (7) that ϕ_1 and its derivatives are required on $y = 0$. We therefore proceed to find the general form of ϕ_1 everywhere in D and, in particular, its value on the free-surface, $y = 0$. We now use equations (14) to write

$$\frac{\partial}{\partial n} G(x,y|x_0,y_0;K) = \frac{\partial}{\partial s} L(x,y|x_0,y_0;K)$$

where L is readily obtained as a series and, on performing an integration by parts of equation (13), it becomes

$$\phi_1(x_0, y_0) = \int_0^l L(x, h(x)|x_0, y_0; K) q_1(x)\, \mathrm{d}x, \qquad (x_0, y_0) \in D \tag{18}$$

after a careful treatment of the discontinuity in the function L. In particular, this equation applies on $y = 0$ and thus provides a means of computing $\phi_1(x_0, 0)$ given the eigenfunction $q_1(x)$.

To make progress with the solution to the boundary-value problem for ϕ_2, we make use of the linearity of the governing equations for ϕ_2, writing $\phi_2 = \xi_1 + \xi_2$ where ξ_i, $i = 1, 2$ both satisfy (2) and (3), whilst ξ_1 satisfies

$$\frac{\partial \xi_1}{\partial y} = 0, \quad \text{on } y = d, \qquad \text{and} \qquad 4K\xi_1 + \frac{\partial \xi_1}{\partial y} = g(x), \quad \text{on } y = 0 \tag{19}$$

and ξ_2 satisfies

$$\frac{\partial \xi_2}{\partial n} = -\frac{\partial \xi_1}{\partial n}, \quad \text{on } y = h(x), \qquad \text{and} \qquad 4K\xi_2 + \frac{\partial \xi_2}{\partial y} = 0, \quad \text{on } y = 0. \tag{20}$$

The decomposition of ϕ_2 into ξ_1 and ξ_2 allows the two sources of complication, namely the FSBC and the bed condition to be divided between two separate problems. The solution for ξ_1 is easily found using separation of variables to be

$$\xi_1 = \sum_{r=0}^{\infty} A_r \cosh \mu_r (d - y) \cos \mu_r x, \qquad \text{where} \qquad \mu_r = r\pi/l \tag{21}$$

for coefficients A_r that are determined by satisfaction of (19*b*). The problem for ξ_2, satisfying the homogeneous boundary condition on $y = 0$ can be approached using Green's Identity with $G(x, y|x_0, y_0; 4K)$ to obtain

$$\xi_2(x_0, y_0) = -\int_C \left(\xi_2(x, y) \frac{\partial G(4K)}{\partial n} + G(4K) \frac{\partial \xi_1}{\partial n} \right) \mathrm{d}s. \tag{22}$$

We follow exactly the same procedure as before, namely differentiating with respect to n_0, integrating by parts and converting to tangential derivatives and apply the bed condition on ξ_2. The bed condition is simplified by noting that

$$\frac{\partial \xi_1}{\partial n} = -\frac{\partial f}{\partial s} \qquad \text{where} \qquad f(x, y) = \sum_{r=1}^{\infty} A_r \sinh \mu_r (d - y) \sin \mu_r x \tag{23}$$

and where, after some manipulation, we may write the integral equation at second-order in the form

$$p(x_0) + \int_0^l L(x, h(x)|x_0, h(x_0); 4K) p'(x)\, \mathrm{d}x = -\int_0^l m(x, x_0; 4K) q_2(x)\, \mathrm{d}x. \tag{24}$$

where $p(x) = f(x, h(x))$. This integral equation for the function $q_2(x)$ relating to a tangential flux along C, has the same kernel as in (17) but operating at $4K$, and inhomogeneous terms that relate through the coefficients A_r in $f(x, h(x))$ to the first order eigenfunction. Once $q_2(x)$ is known, a procedure similar to that at first order can be used to determine ξ_2 everywhere and, in particular,

$$\xi_2(x_0, 0) = \int_0^l L(x, h(x)|x_0, 0; 4K) q_2(x)\, \mathrm{d}x - \int_0^l G(x, h(x)|x_0, 0; 4K) p'(x)\, \mathrm{d}x. \tag{25}$$

This information allows us to be able to calculate the second-order free surface elevation.

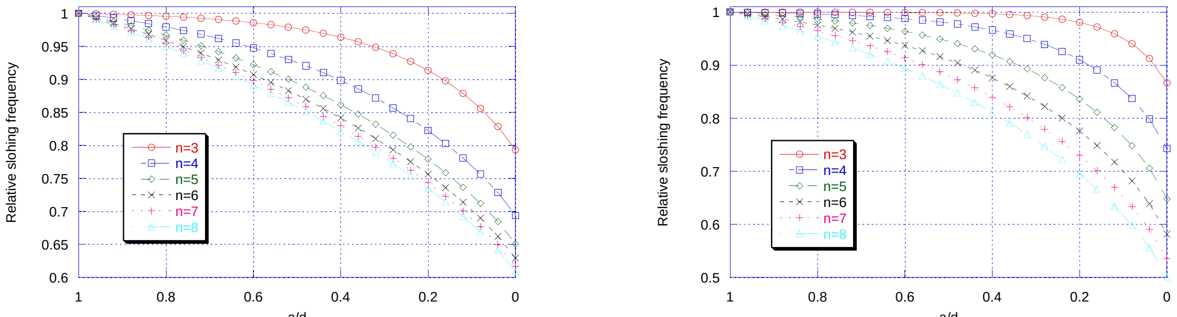
Approximation and numerical method

Although our formulation of the problem so far is exact we must resort to numerical techniques to generate results. We solve the integral equation (17) numerically by using a Galerkin method where we approximate $q_1(x)$ by

$$q_1(x) \simeq \widetilde{q_1}(x) \equiv \sum_{n=1}^{N} a_n v_n(x), \qquad \text{with } v_n(x) = (1/l)\sin(\mu_n x) \text{ wwhere } \mu_n = n\pi/l \tag{26}$$

and $v_n(x)$ are chosen to reflect the anticipated local behaviour at $x = 0, l$. Thus, substituting (26) in (17), multiplying through by $v_m(x_0)$, $m = 1, \ldots, N$ and integrating over $0 < x_0 < l$ results in an $N \times N$ linear system of equations for a_n. Typically, values of $N = 8$ are needed for convergence to six decimal places. On account of the wall conditions, (3) we expand $\phi_1(x, 0)$ in terms of a cosine series

$$\phi_1(x, 0) = \sum_{n=1}^{N} b_n \cos(\mu_n x) \tag{27}$$



(a) Mode 1 (b) Mode 2

Figure 1: Sloshing frequencies for the first two modes (normalised with respect to the flat bed frequency) over a triangular bed making an angle of π/n with the horizontal bed and intercepting the vertical axes at $y = a$

and it turns out that b_n can be efficiently calculated in terms of a_n. Furthermore, it can be shown that

$$A_r = \frac{-i\omega \operatorname{sech} u_r d}{4gl^2(4K - \mu_r \tanh \mu_r d)} \left\{ \sum_{n=0}^{r} b_n b_{r-n}(3K^2l^2 + \pi^2 n(n-2r)) + \frac{\epsilon_r}{2} \sum_{n=0}^{N-r} b_n b_{n+r}(6K^2l^2 + \pi^2(2n^2 + 2nr + l^2)) \right\}$$

the rather simple calculation which represents the point at which non-linearity in the free surface condition is transferred to the second order potential. An identical approximation to (26) is taken for $q_2(x)$ and the Galerkin procedure described above is used to determine the second order solution.

We present results at Figure 1 showing how this approach may be used to calculate sloshing frequencies. It can easily be seen that the frequencies are monotonic decreasing as expected from Fox & Kuttler (1983). It also serves to confirm that the formulation reproduces the known analytical results for the cases $n = 4$ and $n = 6$ for $a = 0$, corresponding to sloshing in triangular canals.

Forced Oscillations

The techniques for free oscillations may be extended to a tank forced at frequencies away from resonance. The situation of interest is where the tank is forced laterally with amplitude a where $a/l = \epsilon \ll 1$ and frequency $\sigma \neq \omega$, the sloshing frequency, where we now write $\Phi_1 = \mathrm{Re}\{\phi_1 \mathrm{e}^{-\mathrm{i}\sigma t}\}$ and consequently obtain a second-order potential composed of steady and double frequency components. For this problem the first and second-order potentials still satisfy Laplace's equation and the FSBC's remain unchanged however the fixed boundary conditions require modification as below

$$\begin{array}{llll} \text{Linear} & \text{2nd Order Steady} & \text{2nd Order } 2\sigma & \\ \dfrac{\partial \phi_1}{\partial x} = \sigma & \dfrac{\partial \Phi_s}{\partial x} = 0 & \dfrac{\partial \phi_2}{\partial x} = 0 & \text{on } x = 0, l \\ \dfrac{\partial \phi_1}{\partial n} = -\mathbf{n}.\nabla \sigma x & \dfrac{\partial \Phi_s}{\partial n} = N_s(x) & \dfrac{\partial \phi_2}{\partial n} = N(x) & \text{on } y = h(x). \end{array} \tag{28}$$

Here $N_s(x)$ and $N(x)$ are the steady and double frequency components respectively of the normal derivative of the second order potential which depend upon ϕ_1 and its derivatives on the bed. Progress may be made if we transform the first order problem using $\phi_1 = x + \varphi_1$ in which case the conditions for φ_1 on the fixed boundaries become homogeneous, as in the free sloshing problem, but at the expense of complicating the FSBC. Nevertheless, the technique outlined in this paper can be used to obtain an inhomogeneous integral equation which is straightfoward to solve for numerically. The transformation of the first-order problem modifies the expressions for $F_s(x)$, $F(x)$, $N_s(x)$ and $N(x)$ in the second-order problem but without introducing additional complexity. Accordingly our methods, including decomposition of the problem, may be applied at second-order. Work on the second-order forced problem is currently in progress and we hope to be able to report results on this problem at the workshop.

References

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Discusser: M. McIver

In your abstract you say that $\varphi_1 \to 0$ as $(x, y) \to (0, h(0)), (L, h(L))$. Surely, this will not be true in general.

Author's reply:

Yes, of course you are correct as $\varphi_1 \to const.$. This was changed in the presentation as it is not crucial to the method. In fact, if you take the limit $x \to 0, 1$ before integrating by parts, the free terms vanish without requiring $\varphi_1 \to 0$.