THE FORM OF THE POTENTIAL OF A HEMISPHERE OSCILLATING IN THE SURFACE OF A FLUID

by

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1 Introduction

In linear water-wave theory the wave potential for a semicircle oscillating on fluid of infinite depth has long been familiar. Thus for a heaving semicircle the potential can be expressed as the sum of an oscillating wave source in the free surface, together with an infinite series of symmetrical wavefree potentials. Similar considerations evidently apply when an arbitrary normal velocity is prescribed on the surface of the semicircle. In general the expansion then includes a horizontal wave dipole and antisymmetric wavefree potentials as well as the symmetrical terms described above.The wave amplitude at infinity is determined by the magnitudes of the wave source and of the wave dipole. For motions without waves at infinity the potential is regular harmonic at infinity.

In the present work we shall consider analogous problems for the hemisphere with its centre in the mean free surface. The wave motion generated by a heaving hemisphere has been treated by [Havelock 1955], the potential is the sum of a wave source in the free surface and of wavefree potentials. Here we shall consider the analytic form of the potential when an arbitrary normal velocity is prescribed on the surface of the hemisphere. Evidently the potential can then be expressed as the sum of Fourier components about the vertical axis of symmetry, and we shall see that the mth Fourier component can be expressed as the sum of the mth order multipole and of the appropriate set of wavefree potentials. Here we shall derive the analytic form of the wave potential. Cartesian axes are taken with the $y-$axis vertical and the origin in the mean free surface . The three-dimensional potential $\phi(x,y,z)\exp(-i\omega t)$ is defined in the region outside the hemisphere ($\mathcal{S}_a^+ : x^2+y^2+z^2=a^2,\ y>0$), where it satisfies

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\phi(x,y,z)=0, \tag{1.1}$$

with the boundary condition

$$K\phi+\frac{\partial\phi}{\partial y}=0 \ \text{ when } y=0,\ x^2+z^2>a^2, \tag{1.2}$$

where $K=\omega^2/g$. It is assumed that at infinity the resulting waves travel outwards. Cylindrical polar coordinates (r,y,α) are taken such that $x=r\cos\alpha$ and $z=r\sin\alpha$, and spherical polar coordinates (R,θ,α) are taken such that $x=R\sin\theta\cos\alpha,\ y=R\cos\theta,\ z=R\sin\theta\sin\alpha$. Let $L>a$ denote any length exceeding the radius of the hemisphere. Our principal result is

Theorem 1 :

$$\phi(x,y,z) \quad = \quad \sum_{m=0}^{\infty}(A_m\cos m\alpha+B_m\sin m\alpha)\ \phi_m(r,y),$$

where

$$\phi_m \quad = \quad \alpha_{0m}L^{m+1}\int_0^\infty \frac{k^{m+1}}{k-K}\,e^{-ky}J_m(kr)\,dk \tag{1.3}$$

$$+ \quad \sum_{\sigma=1}^{\infty}\alpha_{\sigma m}\left((2\sigma+2)\frac{\mathrm{P}^m_{m+2\sigma+2}(\cos\theta)}{R^{m+2\sigma+3}}+K\frac{\mathrm{P}^m_{m+2\sigma+1}(\cos\theta)}{R^{m+2\sigma+2}}\right)L^{m+2\sigma+3}\left(\frac{(2\sigma+2)!}{(2m+2\sigma+2)!}\right)^{1/2}, \tag{1.4}$$

where it is supposed that the path of integration in (1.3) passes below the pole $k = K$. (The same assumption will be made throughout this paper for all integrals with a polar singularity at $k = K$.)

We observe that the wave motion at infinity is contributed by the integral (1.3); when this term is absent the potential at infinity is represented by the wavefree potentials (1.4) and is regular harmonic. We shall also obtain the expansion of the wave term (1.3) near the origin. It will be shown that

$$\begin{aligned}
&\int_0^\infty \frac{k^{m+1}}{k-K}\, e^{-ky} J_m(kr)\, dk \\
= \quad & K^{m+1} \sum_{s=0}^{m} \frac{\mathrm{P}_{m-s}^{-m}(\cos\theta)}{(KR)^{m+1-s}\Gamma(2m+1-s)} + K^{m+1} \sum_{s=0}^{m-1} \frac{(KR)^s \mathrm{P}_s^{-m}(\cos\theta)}{\Gamma(m-s)}. \\
- \quad & 2\pi i K^{m+1} \sum_{N=0}^{\infty} \frac{\partial}{\partial N} \left\{ e^{-\pi i N} \frac{(KR)^{m+N}}{\Gamma(N+1)} \mathrm{P}_{m+N}^{-m}(\cos\theta) \right\}. \qquad (1.5)
\end{aligned}$$

2 Proof of Theorem 1

Only a brief outline can be given here. We begin by considering

$$\Phi_m(R,\theta) = \Phi_m(r,y) = \left(K + \frac{\partial}{\partial y}\right) \phi_m(r,y),$$

for which there is an expansion of the form

$$\Phi_m(R,\theta) = \sum_{s=0}^{\infty} a_{2s} \left(\frac{L}{R}\right)^{m+2s+2} \mathrm{P}_{m+2s+1}^{m}(\cos\theta) \left(\frac{(2s+1)!}{(2m+2s+1)!}\right)^{1/2}. \qquad (2.1)$$

To obtain this expansion, we use the property that $\Phi_m(r,0) = 0$, see (1.2), from which it follows that Φ_m can be continued by Schwarz's Symmetry Principle $\Phi_m(r,y) = -\Phi_m(r,-y)$ into the mirror image of its original region of definition, and is thus defined in the region $(\mathcal{D}_L^{\pm} : x^2 + y^2 + z^2 > L^2, -\infty < y < \infty)$ external to the sphere of radius L. It is evident that $\Phi_m(R,\theta)$ is bounded in the region $\mathcal{D}_L^{\pm}$: the expansion (2.1) follows almost immediately by separation of variables. We can now proceed to the proof of Theorem 1. We note the integral ([Erdélyi 1953], Vol 2, 7.8(10))

$$\Gamma(\nu+m+1)\, \mathrm{P}_\nu^{-m}(\cos\theta) = \int_0^\infty e^{-T\cos\theta} J_m(T\sin\theta)\, T^\nu\, dT, \qquad (2.2)$$

valid when $\nu + m + 1 > 0$, where ([Erdélyi 1953], Vol 1, 3.4(6))

$$\mathrm{P}_\nu^{-m}(\xi) = \frac{1}{\Gamma(m+1)} \left(\frac{1-\xi}{1+\xi}\right)^{\frac{1}{2}m} F(-\nu, 1+\nu;\ m+1;\ \tfrac{1}{2} - \tfrac{1}{2}\,\xi),$$

in the standard notation of hypergeometric functions. It follows that

$$\frac{\mathrm{P}_{m+2s+1}^{m}(\cos\theta)}{R^{m+2s+2}} = \frac{(-1)^m}{(2s+1)!} \int_0^\infty k^{m+2s+1} e^{-ky} J_m(kr)\, dk \text{ when } y > 0. \qquad (2.3)$$

Thus, when $y > 0$, we see that (2.1) is the differential equation

$$\left(K + \frac{\partial}{\partial y}\right) \phi_m(r,y) = (-1)^m \sum_{s=0}^{\infty} \frac{a_{2s} L^{m+2s+2}}{\{(2s+1)!(2m+2s+1)!\}^{1/2}} \int_0^\infty k^{m+2s+1} e^{-ky} J_m(kr)\, dk. \qquad (2.4)$$

When $r > L$, one solution of this equation is given by $\phi_m^\dagger(r, y)$, where

$$(-1)^{m-1}\phi_m^\dagger(r,y) = \sum_{s=0}^{\infty} \frac{a_{2s}L^{m+2s+2}}{\{(2s+1)!(2m+2s+1)!\}^{1/2}} \int_0^\infty k^{m+2s+1} e^{-ky} J_m(kr)\ \frac{dk}{k-K}. \tag{2.5}$$

Since

$$\frac{k^{2s}}{k-K} = \frac{K^{2s}}{k-K} + (k+K)(k^{2s-2} + k^{2s-4}K^2 + \cdots + K^{2s-2}), \tag{2.6}$$

we see that the expansion (1.3)-(1.4) follows at once, when we note that

$$\frac{1}{(2S+1)!} \int_0^\infty (k+K)k^{m+2S+1} e^{-ky} J_m(kr)\ dk \tag{2.7}$$

$$= \quad (-1)^m \left((2S+2)\frac{\mathrm{P}^m_{m+2S+2}(\cos\theta)}{R^{m+2S+3}} + K\frac{\mathrm{P}^m_{m+2S+1}(\cos\theta)}{R^{m+2S+2}} \right) \tag{2.8}$$

from (2.3). It is also easily seen that this expansion is the only solution of (2.4) satisfying the radiation condition at infinity, that (2.8) is a wavefree potential satisfying (1.2) , and that the expansion (1.3) - (1.4) is convergent since the expansion (2.1) is convergent.

It remains to discuss the contribution (1.3), which is a multiple of the integral

$$\int_0^\infty k^{m+1} e^{-ky} J_m(kr)\ \frac{dk}{k-K}, \tag{2.9}$$

where the path of integration passes below the pole $k = K$. By deforming the contour it is not difficult to show that this behaves like a wave for large r. To find the expansion in spherical polar coordinates for small r we write

$$\frac{k^{m+1}}{k-K} \quad = \quad k^m \left(1 + \frac{K}{k} + \frac{K^2}{k^2} + \cdots + \frac{K^{2m}}{k^{2m}}\right) \tag{2.10}$$

$$+ \quad \frac{K^{2m+1}}{k^m(k-K)}. \tag{2.11}$$

We recall the integral (2.2), valid when $\nu + m + 1 > 0$. (Here m is an integer and ν is real.) Thus the finite series (2.10) gives a contribution

$$= \quad K^{m+1} \sum_{s=0}^{m} \frac{\mathrm{P}^{-m}_{m-s}(\cos\theta)}{(KR)^{m+1-s}\Gamma(2m+1-s)} + K^{m+1} \sum_{s=0}^{m-1} \frac{(KR)^s \mathrm{P}^{-m}_s(\cos\theta)}{\Gamma(m-s)}. \tag{2.12}$$

Evidently the integral (2.2) can be transformed into the loop integral

$$(e^{2\pi i\nu} - 1)\Gamma(\nu+m+1)\mathrm{P}^{-m}_\nu(\cos\theta) = \int_\infty^{(0+)} e^{-T\cos\theta} J_m(T\sin\theta)\ T^\nu\ dT, \tag{2.13}$$

where the path of integration starts at ∞, encircles the origin in the positive direction, and ends at $\infty e^{2\pi i}$; this is valid for all real ν. Similarly the term (2.11) gives the integral

$$K^{2m+1} \int_0^\infty e^{-ky} J_m(kr) \frac{dk}{k^m(k-K)} = \frac{1}{2\pi i} K^{2m+1} \int_\infty^{(0+)} e^{-ky} J_m(kr)\ \log\frac{k}{K}\ \frac{dk}{k^m(k-K)}, \tag{2.14}$$

where both branches of the path of integration pass below the pole $k = K$. We now deform the upper branch to pass above $k = K$, this does not change the value of the integral because $\log k/K$ vanishes at $k = K$. We next deform the new path of integration to lie outside the circle $|k| = K$, and thus the expression (2.14) can be written as the series

$$\frac{1}{2\pi i} K^{2m+1} \int_{\infty}^{(0+)} e^{-ky} \frac{J_m(kr)}{k^{m+1}} \log \frac{k}{K} \sum_{N=0}^{\infty} \left(\frac{K}{k}\right)^N dk. \tag{2.15}$$

$$= \frac{1}{2\pi i} K \sum_{N=0}^{\infty} \int_{\infty}^{(0+)} e^{-KRu\cos\theta} J_m(KRu\sin\theta) \log u \frac{du}{u^{N+m+1}} \tag{2.16}$$

$$= \frac{1}{2\pi i} K \sum_{N=0}^{\infty} \frac{\partial}{\partial \nu} \left((e^{2\pi i\nu} - 1)\Gamma(\nu + m + 1) \frac{\mathrm{P}_\nu^{-m}(\cos\theta)}{(KR)^{\nu+1}} \right), \tag{2.17}$$

where $\nu = -N - m - 1$. By using the relations

$$e^{2\pi i\nu} - 1 = 2ie^{\pi i\nu} \sin\pi\nu, \quad \Gamma(Z)\Gamma(1-Z) = \pi/\sin\pi Z \tag{2.18}$$

it is then not difficult to obtain the expansion (1.5). It should be noted that the expansion for the case $m = 0$ can be treated more simply: when the expansion of ϕ_0 along the y-axis is known the complete expansion can be at once inferred. When $m > 0$ the potentials ϕ_m vanish on the y-axis and the simple argument is no longer available.

3 Discussion

In two dimensions it has long been known ([Ursell 1968]) that the velocity potential can be continued into the whole of the upper half-plane outside a large circle, with a cut to infinity along an arbitrary vertical line. For by Schwarz's Symmetry Principle the boundary condition (1.2) gives the continuation of $K\phi + \phi_y$ into the upper half-plane outside the large circle ($\mathcal{C}_L^{\pm} : x^2 + y^2 > L^2$), and integration in the y-direction then gives the continuation of $\phi(x, y)$ into the part of $\mathcal{C}_L^{\pm}$ outside the vertical strip ($-L < x < L$). Then $\phi(x, y)$ can be extended horizontally from this domain into the vertical strip by solving the equation

$$\left(\frac{\partial^2}{\partial x^2} + K^2\right) \phi(x, y) = -\left(\frac{\partial}{\partial y} - K\right) \left(K\phi + \frac{\partial\phi}{\partial y}\right),$$

where the right-hand side is known in the whole of $\mathcal{C}_L^{\pm}$, since $K\phi + \phi_y$ is known in the whole of $\mathcal{C}_L^{\pm}$. This procedure gives $\phi(x, y)$ outside a large circle, with a cut along an arbitrary vertical line. Across the vertical line the discontinuities of the potential and of the normal gradient are found to be simply related to the waves at either infinity; in particular,when there are no waves at infinity the cut is absent, and the point at infinity is then a regular point. (A simpler argument using complex potentials was given earlier in [Ursell 1950].)

In the present note we have treated the corresponding problem in three dimensions. The analytic continuation into the upper half-space by means of (1.2) now excludes a vertical cylinder, but in general the excluded volume cannot be contracted to a single vertical line. This is readily seen if we consider a potential generated by a finite number of wave sources. In the upper half-space this potential is singular (not merely discontinuous) on the vertical lines through the sources, and clearly these singular lines cannot be combined into one singular line. However, when the potential is resolved into Fourier components about a vertical axis, then (as follows from Theorem 1) each Fourier component $\phi_m(R, \cos\theta)\exp(\pm im\alpha)$ can be continued into the upper half-space as far as the axis which is a line of singularities . When there are no waves at infinity the first term (1.3) is therefore absent, and the potential is then regular at the point at infinity and can be expressed as the sum of single-valued spherical harmonics $R^{-n-1}\mathrm{P}_n^m(\cos\theta)\exp(\pm im\alpha)$ outside a large sphere, see(2.8).

References

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