# The influence of a trapped mode on a radiation potential

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## Introduction

In recent years there have been several investigations into the existence of trapped and near-trapped modes, and several calculations have been made to determine the influence of wave-trapping on the forces on an off-shore structure. It is well-known (Linton & Evans[1], Newman[3]) that the force coefficients exhibit a singular behaviour in the vicinity of a near-trapping frequency, but that the types of singularity in the added mass and damping are different. The former singularity has the structure of a simple pole, whilst the singularity in the damping is narrow-banded. The purpose of this work is to calculate the strength and spatial variation of the singular term in the radiation potential for a body which supports an exact trapped mode, and to devise a numerical scheme for the calculation of the first-order correction term to this singularity. It will be shown that there is in fact no singularity in the damping for a body which supports a genuine trapped mode, although as was shown by Newman[3], any small perturbation in the geometry gives rise to singular behaviour in the damping.

## Expansion of a radiation or scattering potential

The determination of either the scattering or radiation potentials for a system of two-dimensional bodies in water of large depth, can be reduced to the solution of the boundary value problem

$$\nabla^2\phi = 0 \quad \text{in the fluid,} \tag{1}$$

$$K\phi - \frac{\partial \phi}{\partial z} = 0 \quad \text{on } z = 0, \tag{2}$$

$$\frac{\partial \phi}{\partial n} = f(x, z, K) \quad \text{on } B, \tag{3}$$

$$\nabla\phi \to 0 \quad \text{as } z \to -\infty, \tag{4}$$

and the radiation condition

$$\frac{\partial \phi}{\partial x} \mp iK\phi \to 0 \quad \text{as } x \to \pm\infty, \tag{5}$$

for the velocity potential $Re[\phi(x, z, K)\,\mathrm{e}^{-\mathrm{i}\omega t}]$. Here $\partial/\partial n$ denotes the derivative in the direction of the outward normal to any bodies $B$ and $f$ arises from either the forced motion of $B$ or the incident wave (which has been subtracted out of the potential). The wave frequency is denoted by $\omega$ and $K = \omega^2/g$, where $g$ is the acceleration due to gravity. Cartesian coordinates $(x, z)$ are chosen, so that the origin is in the mean free surface and the $z$-axis points vertically upwards.

The system of bodies is assumed to support a single trapped mode at wave number $K = K_0$, which means that there is a non-zero solution to the boundary value problem described in (1) - (5) when $f = 0$. This solution, $\phi_0, (x, z)$ must have finite energy and decay as $|x| \to \infty$ and so, without loss of generality, $\phi_0$ may be chosen to be real. A consequence of the Fredholm alternative is that a solution $\phi(x, z, K)$ to a forced problem exists at $K = K_0$ if and only if the forcing is orthogonal to the trapped mode potential $\phi_0$. This orthogonality condition is derived most easily by an application of Greens theorem to $\phi$ and $\phi_0$ for $K \neq K_0$. After some manipulation this yields

$$(K - K_0) \int_{FS} \phi_0(x, 0)\phi(x, 0, K)\,\mathrm{d}x = -\int_B \phi_0(x, z)\, f(x, z, K)\,\mathrm{d}s, \quad K \neq K_0 \tag{6}$$

where $FS$ denotes the mean free surface. If $\phi(x,z,K)$ exists at $K = K_0$ then (6) must hold at that wave number, and the left-hand side of the equation must be zero. However the right-hand side of (6) depends only on the trapped mode potential and the forcing. Thus $\phi(x,z,K)$ exists at $K = K_0$ only if

$$\int_B \phi_0(x,z)\, f(x,z,K_0)\, \mathrm{d}s = 0. \tag{7}$$

It is straightforward to show that this orthogonality condition is always satisfied if $f$ is the forcing due to an incident wave. However the condition is not in general true if $f$ is the forcing in a radiation problem. For example (7) is not true if $f = n_z$, the forcing in the heave problem for the bodies considered by McIver[2], but it is true if the forcing arises from any antisymmetric problem for that geometry, such as the sway or roll problems. Thus the scattering, sway and roll potentials exist for that geometric configuration at $K = K_0$, but the heave potential does not.

In the case where (7) is not satisfied, the form of (6) suggests the following Laurent expansion for $\phi(x,z,K)$ near $K = K_0$,

$$\phi(x,z,K) = \frac{\psi(x,z)}{K-K_0} + \phi_1(x,z) + O(K-K_0), \quad \text{as} \quad K \to K_0. \tag{8}$$

This expansion is substituted into the governing boundary value problem for $\phi$ and the equations are expanded in powers of $K - K_0$ and like terms are equated. (Care must be taken with this procedure as the expansion is non-uniform as $|x| \to \infty$. However there is no problem for terms up to $O(1)$ in the expansion.) As the forcing on the body is $O(1)$ and the leading order term in $\phi(x,z,K)$ is $O((K-K_0)^{-1})$, $\psi(x,z)$ satisfies a zero Neumann condition on $B$ and hence satisfies the homogeneous boundary value problem at wave number $K = K_0$. Thus $\psi$ is a multiple of the trapped mode potential with the multiplicative factor, $A$, determined by equating powers of $K - K_0$ in (6). This yields

$$\psi(x,z) = A\phi_0(x,z), \quad \text{where} \quad A = -\int_B \phi_0(x,z) f(x,z,K_0)\, \mathrm{d}s \Big/ \int_{FS} \phi_0^2(x,0)\, \mathrm{d}x \tag{9}$$

(It may be shown that no other type of singular term is possible in $\phi(x,z,K)$, as such a term would have to satisfy the homogeneous boundary value problem and also be orthogonal to $\phi_0$.) The form of $A$ makes it clear that there is no singular term in the scattering potential or any other potential which satisfies (7), as $A = 0$ for such a potential.

The first-order correction potential satisfies

$$\nabla^2\phi_1 = 0 \quad \text{in the fluid,} \tag{10}$$

$$K_0\phi_1 - \frac{\partial\phi_1}{\partial z} = -A\phi_0(x,0) \quad \text{on } z = 0, \tag{11}$$

$$\frac{\partial\phi_1}{\partial n} = f(x,z,K_0) \quad \text{on } B, \tag{12}$$

$$\nabla\phi_1 \to 0 \quad \text{as } z \to -\infty, \tag{13}$$

and the radiation condition

$$\frac{\partial\phi_1}{\partial x} \mp iK_0\phi_1 \to 0 \quad \text{as } x \to \pm\infty. \tag{14}$$

This is a forced problem at the trapped mode frequency and by construction the solution may be shown to exist. The Fredholm Alternative guarantees that $\phi_1$ is unique once the condition which arises from (6),

$$\int_{FS} \phi_1(x,0)\phi_0(x,0)\, \mathrm{d}x = \int_B \phi_0(x,z)\frac{\partial f}{\partial K}(x,z,K_0)\, \mathrm{d}s \tag{15}$$

is applied. Note that the right-hand side of (15) is zero if the $f$ is the forcing which arises in a radiation problem.

## Numerical method

The boundary value problem in (10) - (14) with the side condition (15) is solved using a boundary integral equation technique. An application of Green's theorem to $\phi_1$ and the two-dimensional, infinite depth Green's function, $G(x, z; \xi, \eta)$ which satisfies $G \sim \ln[(x-\xi)^2 + (z-\eta)^2]/4\pi$ as $(x-\xi)^2 + (z-\eta)^2 \to 0$, yields an integral equation for $\phi_1$ around the boundary of the body, namely

$$\frac{1}{2}\phi_1(x,z) - \int_B \phi_1(\xi,\eta)\frac{\partial G}{\partial n_{(\xi,\eta)}}(x,z;\xi,\eta)\,\mathrm{d}s = -\int_B G(x,z;\xi,\eta)f(\xi,\eta)\,\mathrm{d}s - 2AF(x,z), \quad (x,z) \in B, \tag{16}$$

and the side condition becomes

$$\int_B \phi_1(x,z)\frac{\partial F}{\partial n} - F(x,z)f(x,z)\,\mathrm{d}s - A\int_{FS} F(x,0)\phi_0(x,0)\,\mathrm{d}x = 0, \tag{17}$$

where

$$F(x,z) = \int_{FS} \phi_0(\xi,0)G(x,z;\xi,0)\,\mathrm{d}\xi. \tag{18}$$

The bodies are discretised into $n$ elements and a piecewise constant approximation to the complex potential $\phi_1$ is used to transform (16) and (17) into $2n+2$ equations in $2n$ unknowns. The original equations have a unique solution and so a NAG routine is applied to the discretised equations to find the least squares approximation to this solution.

## Results and discussion

Calculations were made for the pair of bodies considered by McIver[2], with 65 panels used on each body. In this case the trapped mode potential is known explicitly and so some parts of the integrals in $F(x,z)$ were done analytically. The non-dimensional heave added mass and damping are given by

$$\frac{a_{33}}{\rho l^2} = -\frac{1}{Kl - K_0 l}\frac{\left[\frac{1}{l}\int_B \phi_0(x,z)n_z\,\mathrm{d}s\right]^2}{\left[\frac{1}{l}\int_{FS} \phi_0^2(x,0)\,\mathrm{d}x\right]} + Re\left[\frac{1}{l^2}\int_B \phi_1(x,z)n_z\,\mathrm{d}s\right] + O(Kl - K_0 l) \tag{19}$$

and

$$\frac{b_{33}}{\rho\omega l^2} = Im\left[\frac{1}{l^2}\int_B \phi_1(x,z)n_z\,\mathrm{d}s\right] + O(Kl - K_0 l) \tag{20}$$

respectively, where the length scale used in the non-dimensionalisation is chosen so that $K_0 l = 1$. It is clear that the added mass has a singularity at the trapped mode frequency with a strength that depends only on the trapped mode potential but the damping is bounded and, from numerical calculations, is non-zero. Figure 1 show a comparison between the asymptotics for the heave added mass and damping and a direct numerical calculation using a standard boundary integral equation technique with the same number of panels on each body and with irregular frequencies removed. In the second graph $K_0 l$ is taken to be 1 for the scaling of the asymptotics, but a numerical estimate of $K_0 l$ is used to scale the direct numerical calculations. This estimate is obtained by taking a weighted average of the positions of the wave numbers of the largest positive and negative values of the added mass.

As expected, the direct numerical calculations fail very close to $K_0 l = 1$ but nonetheless agree with the asymptotics of the added mass in the vicinity of this frequency. However, although there is no singularity in the asymptotics for the damping there is clearly some singular behaviour in the numerical calculations of this quantity in the neighbourhood of the trapped mode frequency. This behaviour was explained by Newman[3] by noting that any numerical discretisation of the

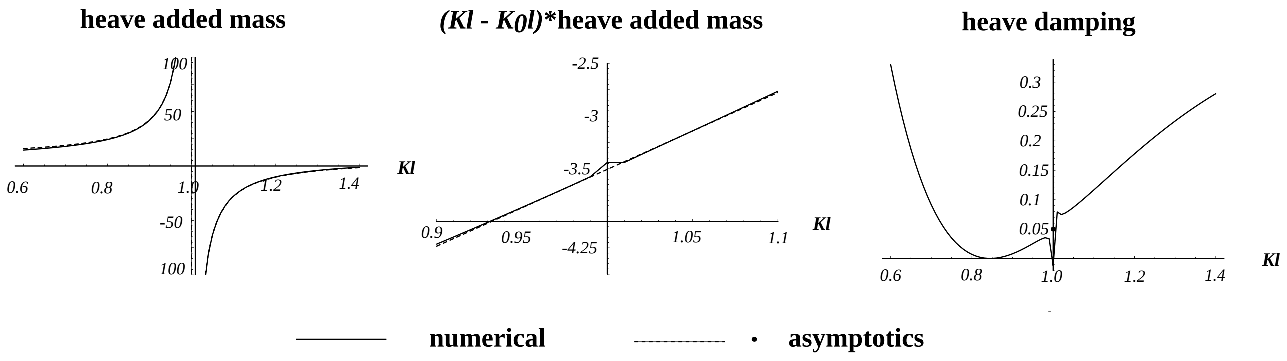


Figure 1: Heave added mass and damping for bodies which support a trapped mode.

equations or geometry has the effect of transforming a trapped mode into a near-trapped mode. This means that whereas the complex force coefficient for bodies which support a genuine trapped mode has poles on the real frequency axis at $\omega = \pm\omega_0$, the poles lie off the axis for the discretised body. Some simple algebra shows that the damping for the approximated body has the form

$$\frac{b_{33}}{\rho\omega l^2} \sim \frac{B_0\epsilon}{(Kl - K_0 l)^2 + \epsilon^2} + B_1 \to B_0\pi\delta(Kl - K_0 l) + B_1 \quad \text{as} \quad \epsilon \to 0, \tag{21}$$

where $B_0$ and $B_1$ are constants, $\epsilon$ is a small parameter which measures the perturbation of the body and $\delta$ is the Dirac delta function. Thus, no matter how accurate is the discretisation of the geometry and equations, it is impossible to totally eliminate all the singular behaviour in the damping.

## Conclusion

An analytic expression for the leading order behaviour of a radiation potential near a trapped mode frequency has been obtained and the next approximation has been determined numerically. The analysis confirms the results of Linton & Evans[1] and Newman[3] and shows that there is a simple pole structure in the added mass and that, although there is no singularity in the damping for a body which supports a genuine trapped mode, there is a narrow banded, delta function like singularity in the damping for any small perturbation of such a body.

## Acknowledgement

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## References

[1] Linton, C. M. & Evans, D. V. 1992 The radiation and scattering of surface waves by a vertical circular cylinder in a channel. *Phil. Trans. R. Soc. Lond.* A **338**, 325–357.

[2] McIver, M. Trapped modes supported by submerged obstacles. *Proceedings of the Royal Society of London* A, **456** 1851–1860, 2000.

[3] Newman, J. N. 1999 Radiation and diffraction analysis of the McIver toroid. *J. Engng. Math.* **35**, 135–147.

**Question by** : R. Porter
All of your results are for a symmetric trapped mode. How would your results be affected if you were to consider a trapping structure which supports an antisymmetric trapped mode?

**Author's reply**:
The same analysis could be done if the trapped modes were antisymmetric, although in this case there would be singular behaviour in the antisymmetric radiation potentials, sway and roll.

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**Question by** : J.N. Newman

1. Is there any way to explain the difference between the radiation and scattering problem, aside from the orthogonality relation (7)?
2. In your talk you referred to the "reciprocity" or Haskind relation, between the b33 and the exciting force. Is this relation still valid here, given that its derivation involves the radiation potential?

**Author's reply**:

1. I do not know of any simple argument
2. The Haskind relation should be valid in the limit as the trapped mode frequency is approached, but not at the trapped mode frequency itself. However there is an issue here which is related to the first question which I still haven't fully understood. The scattering potential exists at the trapped mode frequency and neighbouring frequencies so it may be argued that the vertical exciting force should also exist and be defined at the trapped mode frequency, although the damping does not exist.

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**Question by** : M. Kashiwagi
The asymptotic expression for $b_{33}$ given as eq.(21) in the abstract shows an even function with respect to $K=K_0$, but the numerical result shown in Fig.1 shows an odd function behaviour around $K=K_0$. What is the reason of this difference?

**Author's reply**:
Only the leading order term in the damping has been obtained so it doesn't say anything about the behaviour of the damping apart from its actual value at the trapped mode frequency itself.

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**Question by** : M. Meylan
Added mass and damping are the real and imaginary parts of the same function. Could you extend the theory to near trapped modes?

**Author's reply**:
Possibly although it would be more complicated. As you said in the discussion one would need to look at the adjoint operator. I think that this is worth exploring further.

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**Question by** : D.V. Evans
The surge problem <u>does</u> have a solution at $K_0$ the trapped mode wavenumber. Do you get a similar behaviour near K˜K0 as described in your paper?

**Comment by** : D.V. Evans

You assume the trapped mode potential is unique at $K_0$. There do exist situations in which this is not the case as Shipway has shown for two concentric cylindrical shells, but it seems unlikely to be the case in your example in 2D.

**Author's reply**:

1. There is no singular behaviour in the antisymmetric radiation potentials at a symmetric trapped mode potential.
2. You could apply Green's theorem to a radiation potential and each of the trapped mode potentials in turn to get 2 sets of orthogonality conditions. I would then expect the leading order behaviour of the radiation to still have a pole at the trapped mode frequency but for the spatial variation to be a linear combination of both trapped mode potentials.