Deflection of a Very Large Floating Platform with variable elastic properties

A. J. Hermans

TU Delft, Faculty of ITS, Department of Applied Mathematical Analysis, Mekelweg 4, 2628 CD Delft, The Netherlands, email: a.j.hermans@its.tudelft.nl

1. Introduction

At the 16th IWWWFB we presented results for the deformation of a VLFP with constant elastic parameters. The short-wave theory we presented made use of an integral-differential equation. The theory is extended for finite water depth making use of a special choice of the Green's function. In this presentation we apply the *ray* method to the inhomogeneous case. The constant parameter case serves as a *canonical* problem to generate the edge or'initial' conditions. Numerical results are shown.

2. Mathematical formulation

The fluid is incompressible, so we introduce the velocity potential $V(x,t) = \nabla\Phi(x,t)$, where $V(x,t)$ is the fluid velocity vector. We assume waves in still water. Hence $\Phi(x,t)$ is a solution of the Laplace equation

$$\Delta\Phi = 0 \quad \text{in the fluid,} \tag{1}$$

together with the linearised kinematic condition, $\Phi_z = w_t$, and dynamic condition, $p/\rho = -\Phi_t - gw$, at the linearized free water surface $z = 0$, where $w(x,y,t)$ denotes the free surface elevation, and ρ is the density of the water. The linearised free surface condition outside the platform becomes:

$$\frac{\partial^2\Phi}{\partial t^2} + g\frac{\partial\Phi}{\partial z} = 0 \quad \text{at } z = 0 \text{ and } (x,y) \in \mathcal{F}, \tag{2}$$

The platform is assumed to be a thin layer at the free-surface $z = 0$, which seems to be a good model for a shallow draft platform. The platform is modelled as an elastic plate with zero thickness. To describe the deflection $w(x,y)$ we apply the isotropic thin plate theory, which leads to an equation for w of the form:

$$m(x,y)\frac{\partial^2 w}{\partial t^2} = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(D(x,y)\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)\right) + p|_{z=0}, \tag{3}$$

where $m(x,y)$ is the mass of unit area of the platform while $D(x,y)$ is its equivalent flexural rigidity. We apply the operator $\frac{\partial}{\partial t}$ to (3) and use the kinematic and dynamic condition to arrive at the following equation for Φ at $z = 0$ and in the platform area $(x,y) \in \mathcal{P}$:

$$\left\{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{D(x,y)}{\rho g}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right) + \frac{m(x,y)}{\rho g}\frac{\partial^2}{\partial t^2} + 1\right\}\frac{\partial\Phi}{\partial z} + \frac{1}{g}\left\{\frac{\partial^2}{\partial t^2}\right\}\Phi = 0\,. \tag{4}$$

The free edges of the platform are free of shear forces and moment. We assume that the radius of curvature, in the horizontal plane, of the edge is large. Hence, the edge may be considered to be straight locally. We then approximate the boundary conditions at the edge by:

$$\frac{\partial^2 w}{\partial n^2} + \nu\frac{\partial^2 w}{\partial s^2} = 0 \;\text{ and }\; \frac{\partial^3 w}{\partial n^3} + (2-\nu)\frac{\partial^3 w}{\partial n \partial s^2} = 0, \tag{5}$$

where ν is Poisson's ratio, n is in the normal direction, in the horizontal plane, along the edge and s denotes the arc-length along the edge. At the bottom of the fluid region $z = -h$ we have

$$\frac{\partial \Phi}{\partial z} = 0. \tag{6}$$

The harmonic wave can be written as $\Phi(x,t) = \phi(x)\ \mathrm{e}^{-\mathrm{i}\omega t}$. Due to the large length scales and elastic parameters involved we introduce dimesionless coordinates and parameters in the following way:

$$x' = \frac{x}{L},\ h' = \frac{h}{L},\ K = \frac{\omega^2 L}{g},\ \mu = \frac{m\omega^2}{\rho g},\ \mathcal{D} = \frac{DK^4}{L^4 \rho g}$$

. The parameters μ and $\mathcal{D}$ are of order one for large values of K. In a practical situation, where L is of the order of 1000 meter and a normal sea spectrum, this is the case. After dropping the primes we obtain at $z = 0$

$$\left\{ \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \left(\frac{\mathcal{D}(x,y)}{K^4} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \right) - \mu(x,y) + 1 \right\} \frac{\partial \phi}{\partial z} - K\phi = 0\ . \tag{7}$$

The undisturbed incident wave equals

$$\phi^{inc}(x) = \frac{g\zeta_\infty}{\mathrm{i}\omega} \frac{\cosh(k_0(z+h))}{\cosh(k_0 h)} \exp\{\mathrm{i}k_0(x\cos\beta + y\sin\beta)\}, \tag{8}$$

where ζ_∞ is the wave height, ω the frequency, while the wave number obeys the dispersion relation, $k_0 \tanh(k_0 h) = K$, for finite water depth. We contimue with the deep water case $h = \infty$, hence $K = k_0 = \omega^2 L/g$, and we assume that the potential underneath the plate can be written as a superposition of *ray*-mode solutions as follows:

$$\phi(x,K) = \sum_n \alpha_n(x,K)\ \mathrm{e}^{\mathrm{i}KS_n(x)}, \tag{9}$$

where $S_n(x)$ is the phase function and $\alpha_n(x,K)$ the amplitude function of the n^{th} mode. In (9) each mode is written as a regular series expansion with respect to inverse powers of iK,

$$\alpha_n(x,K) = \sum_{j=0}^{N} \frac{\alpha_{n,j}(x)}{(\mathrm{i}K)^j} + o((\mathrm{i}K)^{-N}). \tag{10}$$

We now drop the index n of the mode for a while. Insertion of (9) into the Laplace equation (1) gives

$$-K^2 \alpha \nabla_3 S \cdot \nabla_3 S + \mathrm{i}K(2\nabla_3 \alpha \cdot \nabla_3 S + \alpha \Delta_3 S) + O(1) = 0. \tag{11}$$

The subscript 3 is used to indicate the three-dimensional ∇ and Δ operators. If no subscripts are used the operators are two-dimensional in the horizontal plane. Next we insert (10) and compare orders of magnitude in (11). This leads to a set of equations for S and α_0 to be satisfied in the fluid region:

$$O(K^2) \quad : \quad \nabla_3 S \cdot \nabla_3 S = 0, \tag{12}$$

$$O(K^1) \quad : \quad 2\nabla_3 \alpha_0 \cdot \nabla_3 S + \alpha_0 \Delta_3 S = 0. \tag{13}$$

We now insert (9) into the condition at $z = 0$ (7). The first two terms in the expansion become

$$O(K^1) : \left\{ \mathcal{D}(x,y)(S_x^2 + S_y^2)^2 - \mu(x,y) + 1 \right\} \mathrm{i}S_z = 1 \tag{14}$$

and $O(K^0)$:

$$\alpha_0\mathcal{D}\left[\frac{\partial}{\partial z}(S_x^2+S_y^2)^2+2S_z\left\{\frac{\partial}{\partial x}S_x(S_x^2+S_y^2)+\frac{\partial}{\partial y}S_y(S_x^2+S_y^2)\right\}\right]+ \tag{15}$$
$$\alpha_{0z}\left\{\mathcal{D}(x,y)(S_x^2+S_y^2)^2-\mu(x,y)+1\right\}+(4\mathcal{D}\nabla\alpha_0\cdot\nabla S+2\alpha\nabla\mathcal{D}\cdot\nabla S)S_z(S_x^2+S_y^2)=0.$$

If we write $r=\mathrm{i}S_z$ and combine (12) with (14 we obtain the dispersion relation at $z=0$

$$(\mathcal{D}(x,y)r^4-\mu(x,y)+1)r=1, \tag{16}$$

combined with

$$S_x^2+S_y^2=r^2. \tag{17}$$

The last equation has the same form as the well known *eikonal* equation in geometrical optics, however in this case the right-hand side is given by an implicit relation (16). In the case of constant elastic coefficients r is a constant and the ray are straight lines as may be expected. We assume that there is a propagating wave solution with a real-valued phase function S. The characteristics (*rays*) become:

$$\frac{\mathrm{d}x}{\mathrm{d}\sigma}=\mathcal{F}S_x \quad,\quad \frac{\mathrm{d}y}{\mathrm{d}\sigma}=\mathcal{F}S_y \quad,$$
$$\frac{\mathrm{d}S_x}{\mathrm{d}\sigma}=-\mathcal{D}_x r^5 \quad,\quad \frac{\mathrm{d}S_y}{\mathrm{d}\sigma}=-\mathcal{D}_y r^5 \quad,\quad \frac{\mathrm{d}S}{\mathrm{d}\sigma}=\mathcal{F}r^2, \tag{18}$$

with $\mathcal{F}=(5\mathcal{D}r^4-\mu+1)/r$ and σ the parameter along the ray.

To obtain an equation for the amplitude α_0, at $z=0$, we use the equation in the fluid (13) to eliminate the z-derivatives in (15). The second order derivative S_{zz} is obtained by means of differentiation with respect to z of (12). We finally get for variable $\mathcal{D}$ and constant μ

$$\frac{\mathrm{d}\alpha_0}{\mathrm{d}\sigma}=-\alpha_0 M\{S\}, \tag{19}$$

where the operator $M\{S\}$ is defined as:

$$M\{S\}(x,y)=\frac{\frac{\mathrm{d}r}{\mathrm{d}\sigma}\left\{16r^4\mathcal{D}-\frac{1}{r}\right\}-\mathcal{F}(S_{xx}+S_{yy})(4\mathcal{D}r^5+1)+4r^4\frac{\partial\mathcal{D}}{\partial\sigma}}{8\mathcal{D}r^5+2}. \tag{20}$$

In principle we can solve these equations if initial conditions for the wave-modes are available. We have one travelling wave-mode and two evanescent modes. The problem, however, is that we can not derive a set of initial conditions for the amplitudes. One should think of writing the field outside the platform as a superposition of an incident and a reflected wave. So we have four unknown coefficient to determine, while there are only two conditions at the edge of the plate. One may try to impose some matching conditions, such as continuity of velocity and potential. This kind of conditions hold in the fluid domain and not at the $z=0$ only. The fact that the z-dependency of the potentials described are different underneath the platform and outside the platform make it impossible to succeed in matching the two fields.

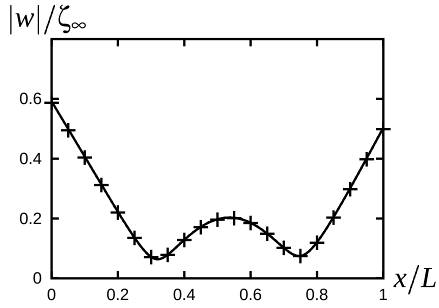
3. Initial conditions and results

We restrict ourselves to the case that the waves are perpendicular to the edge of the platform. For the case of deep water it can be shown that three modes at the plate are sufficient to receive accurate results. If the depth becomes smaller more modes can be taken into account.

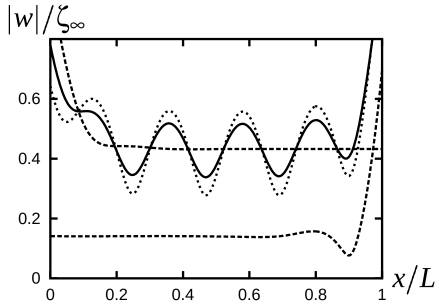
Two conditions, for the amplitude functions, at the edge of the platform follow directly from the boundary conditions at the edge (5), while the third one becomes

$$\sum_{n=1}^{3} \frac{K(\mathcal{D}(0)S_{\xi}^{4} - \mu(0))}{k_0 - KS_{\xi}} a_n = \zeta_{\infty}. \tag{21}$$

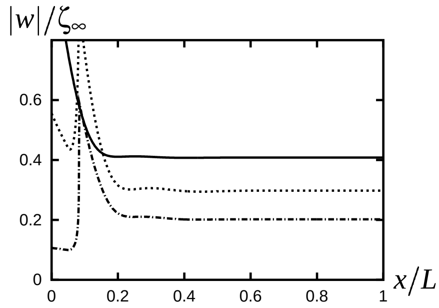
These boundary equation follow from an analytic evaluation of the differential-integral formulation described earlier, see [1,2]. First we show a comparison with results of Takagi et al [3] for a finite strip with constant coefficients, and the effect of reflection and multipe reflection for a similar case. The results of Takagi are obtained by a different asymptotic method, no differences can be distinguished in the figure. In the next two figures results for a semi-infinite plate are shown. The rigidity is varied over an finite interval by means of a continuous (sine-) function.



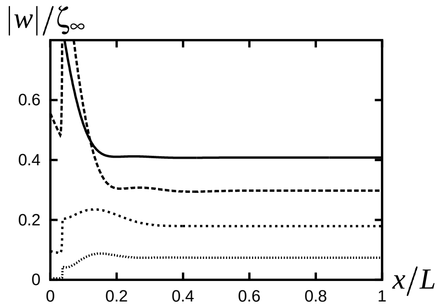
(a) Comparison with Takagi



(b) Multiple reflection $\lambda_0/L = 0.3$.



(c) Variation of rigidity



(d) Variation of rigidity and mass

References

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2. HERMANS A.J., (2002). Geometrical-Optics for the Deflection of a Very Large Floating Flexible Platform. *Applied Ocean Research*, vol. , 8 pages.
3. TAKAGI K., SHIMADA K and IKEBUCHI Y (2000). An anti-motion device for a large floating structure, *Marine Structures*, vol. 13, p. 421-436.

Question by : T. Miloh
I thought that the ray method is rather limited for very short waves and for this reason I was impressed by the almost perfect agreement with Takagi's data that you presented which show (surprisingly ??) that the ray method can be used for much longer wave lengths.

Author's reply:
The ray method sometimes gives rather accurate results for shorter waves than expected. In this case it is expected that the asymptotic results may be rather close to the exact values. In my opinion this is due to the fact that the case we consider has a very simple wave structure (plain waves). The length scale involved is the length of the interval where the coefficients change. The comparison with the results of Takagi does not concern the ray method results, but the method to solve the canonical problem to obtain the *missing* initial condition for the ray solution. The method consists of a superposition of exponential functions describing the exact solution, so it is not so surprising that these results coincide so well. For the asymptotic results for the inhomogeneous problem no data obtained by other methods are available.

Question by : K. Takagi
Is your method applicable for a real VLFS, which has a jump of the rigidity, without any difficulty ?

Author's reply:
The asymptotic ray method is not applicable directly to this problem, due to the fact that implicitely I used the length of the interval where the coefficients change as length scale. However, the original method used to compute the initial conditions (the canonical problem) can be extended to solve this problem exactly. The two regions may be connected in several ways, if we make a rigid connection we may employ in the 2-D case continuity of $w(x)$ and its first three derivatives or for beam seas the Lamé constants may be varied or one may keep the connected plates free of each other. The missing boundary conditions (three in this case) in all these cases are obtained as before. There is no need to match three eigenmode expansions. The formulation of the integral equation guarantees continuity of the velocity potential and its derivatives. I will propose to the organisers of next workshop an abstract with, among others, results of this extension.