Two-layer flows generated by moving obstacles

J.–M. VANDEN-BROECK[a], F. DIAS[b]

[a] *School of Mathematics, University of East Anglia, Norwich NR4 7TJ, UK* [b] *Centre de Mathématiques et de Leurs Applications, Ecole Normale Supérieure de Cachan, 61 avenue du Président Wilson, 94235 Cachan cedex, France*

Abstract

Nonlinear waves in a forced channel flow of two contiguous homogeneous fluids of different densities are considered. The total depth is finite. The forcing is due to a bottom obstruction. The study is restricted to steady two dimensional flows. A weakly nonlinear analysis is performed. At leading order the problem reduces to a forced Korteweg–de Vries equation, which can be integrated exactly. Near a critical value of the ratio of layer depths, the forced Korteweg–de Vries equation is no longer valid and a forced modified Korteweg–de Vries equation must be used. The weakly nonlinear results are validated by comparison with numerical results based on the full governing equations. The numerical method is based on boundary integral equation techniques. A new family of solutions with a train of downstream waves is found. Results for a single fluid layer are also presented.

The theory of stratified flows past submerged obstacles is a classical subject of fluid mechanics. The monograph by Baines (1995) is devoted to this rich topic. In this talk we discuss motion forced by obstacles in the flow of two contiguous homogeneous fluids of different densities. The total depth is finite and the flow two dimensional and steady.

Such configurations allow a wide variety of possible flow configurations. For example, in the case of a single fluid layer flowing over a semi-circular obstacle, four different flow scenarios are known to be possible, depending upon the upstream Froude number F, which is the ratio of the upstream uniform velocity to the critical speed of shallow water waves. There are two important values of the Froude number, $F_C > 1$ and $F_L < 1$, such that: (i) when $F < F_L$, there is a unique downstream wave matched with the upstream (subcritical) uniform flow; (ii) when $F = F_L$, the period of the wave extends to infinity and the solution becomes a hydraulic fall (conjugate flow solution) – the flow is subcritical upstream and supercritical downstream; (iii) when $F > F_C$, there are two symmetric solitary waves sustained over the site of forcing, and at $F = F_C$ the two solitary waves merge into one; (iv) when $F > F_C$, there is also a one-parameter family of solutions matching the upstream (supercritical) uniform flow with a wave downstream. Note that for a particular value of $F > F_C$, the downstream wave can be eliminated and the solution becomes a reversed hydraulic fall, i.e. the same as solution (ii) except that the flow is reversed. All types of flows have been computed numerically. See for example Forbes & Schwartz (1982) for type (i), Forbes (1988) for type (ii), Vanden–Broeck (1988) for types (ii) and (iii), Dias & Vanden–Broeck (1989) for types (ii) and (iii), Dias & Vanden–Broeck (2001) for type (iv). Similar flows occur if the obstacle is on the free surface. See for example the paper by Asavanant & Vanden–Broeck (1994) on flows past a surface-piercing object.

Our purpose is to provide a generalization to the situation in which two contiguous homogeneous fluids of different densities flow over an obstacle. The upper boundary is assumed to be rigid.

Interfacial waves in the absence of obstacle have been studied analytically, numerically and experimentally. There is an extensive literature on the subject. In the presence of an obstacle, there are less results available. Sha & Vanden–Broeck (1993) solved numerically an integrodifferential equation to compute two-layers flows past a semicircular obstruction. Forbes (1989) generalized his results on one-layer hydraulic falls (Forbes 1988) to a two-layer configuration. He considered free-surface boundary conditions. In other words the top boundary is free. Following Forbes (1989), we choose the obstacle to be a semi-circular cylinder but we only consider rigid tops. In other words the top boundary is a solid wall. Results similar to those presented can be obtained with different obstacle shapes. Shen (1992) derived forced Korteweg–de Vries equations for long nonlinear waves in two-layer flows with a free surface and compared his results with those of Forbes (1989).

The velocities of the uniform flows far upstream are assumed to be the same and are denoted by U. The heights of the uniform flow far upstream are denoted by h_1^* (bottom layer) and h_2^* (upper layer). All along, quantities related to the upper layer of fluid will have the subscript 2, while those related to the lower layer will be indexed with 1. The Froude number upstream is defined by

$$F = \frac{U}{c_0}, \quad c_0 = \left[g(1-\rho) h_1^* \left(\frac{h_2^*}{h_2^* + \rho h_1^*} \right) \right]^{1/2}, \tag{0.1}$$

where g is the acceleration due to gravity and ρ the density ratio ($\rho < 1$). The speed c_0 comes from linear theory: If $F < 1$, the flow is subcritical, while the flow is supercritical if $F > 1$. In the one-layer case ($\rho = 0$), the Froude number reduces to the classical Froude number $F = U/\sqrt{gh_1^*}$. In the case of two fluids with almost the same density ($\rho \approx 1$), the so-called Boussinesq limit,

$$c_0 \approx \left[g(1-\rho) h_1^* h_2^* / (h_1^* + h_2^*) \right]^{1/2}.$$

A weakly linear analysis is performed for subcritical as well as supercritical flows, away from the critical thickness ratio. This is based on the so-called forced Korteweg–de Vries equation (fKdV):

$$\frac{d^{*2}}{6h_1h_2} \eta^*_{x^*x^*x^*} + \frac{3}{2d^*}(h_2 - h_1)\eta^* \eta^*_{x^*} - (F-1)\eta^*_{x^*} = -\frac{h_2}{2} h^*_{x^*}, \tag{0.2}$$

where

$$d^* = h_1 h_2 (h_1^* + h_2^*).$$

Here $y = \eta^*(x^*)$ is the equation of the interface and $y = h^*(x^*)$ is the equation of the bottom.

If $2|\eta^*/d^*| \gg (h_2 - h_1)$, (0.1) is no longer valid and the following forced modified Korteweg–de Vries equation is derived:

$$\frac{d^{*2}}{6h_1h_2} \eta^*_{x^*x^*x^*} + \frac{3}{2d^*} \left[(h_2 - h_1)\eta^* - 2\frac{h_1^3 + h_2^3}{d^*} \eta^{*2} \right] \eta^*_{x^*} - (F-1)\eta^*_{x^*} = -\frac{h_2}{2} h^*_{x^*}. \tag{0.3}$$

We show that the weakly nonlinear theories are particularly useful for identifying the various types of solutions and the corresponding numbers of independent parameters. A numerical scheme, based on integro differential equations, is described to solve the fully nonlinear problems. The numerical procedure follows closely the work of Sha and Vanden-Broeck (1993). Sha and Vanden-Broeck restrict their attention to solutions which are symmetric with respect to the y-axis. Here we relax this restriction in order to be able to compute fronts and generalised fronts. Numerical solutions are presented and new types of solutions with downstream waves are discussed.

References

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Discussion Sheet

Abstract Title :	Two-Layer		
(Or) Proceedings Paper No. :	49	***Page :***	191
First Author :	Vanden-Broeck, J.M.		
Discusser :	John Grue		

Questions / Comments :

I am glad to see your results on the limiting configurations of solitary waves in two-layer fluid. They fit with our recent computations (Rusas and Grue, 2002).
Could you comment on the performance of your method for $\Delta\rho/\rho \ll 1$, but still not taking the Boussinesq limit?
A second question: In our simulations of transcritical two-layers flows at an obstacle we found a considerable discrepancy between the full equations and the KdV equation (Grue et.al. 1997, JFM). This was the case even for weak forcing, except when there was no effects, i.e. the forcing becasme too weak.
Could you comment on you use of the KdV equation?

Author's Reply :
(If Available)

We are also very pleased with the agreement between our numerical results and those of Rusas and Grue (2002)(Eur. J. Mech. B/Fluids 21, 185). Our method performs well for $\Delta\rho/\rho \ll 1$ (i.e. $\Delta\rho/\rho = 0.1$ or $\Delta\rho/\rho = 0.01$).
We only used the weekly nonlinear theory as a guide to find the appropriate number of parameters and the qualitative shape of the interface.

Questions from the floor included; Andy King, Touvia Miloh & Marshall Tulin.