Sloshing modes in a cylindrical tank containing multiple cylinders

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1. Introduction

In this paper we describe a method for determining the sloshing modes in a uniform cylindrical tank containing multiple vertical circular cylinders. We use the fact that it is possible to exploit the circular geometry of the tank and each of the elements within the tank to express the solution of the problem exactly. The key ingredient that permits this progress is Graf's addition theorem for Bessel functions which has been applied to many problems involving multiple circular cylinders. From these, most relevant to the present problem is the application of Graf's theorem used by Linton & Evans (1990) to study the interaction of a parallel crested incident wave field with an arbitrary array of uniform vertical cylinders. The solution of our problem is expressed in terms of an infinite system of equations whose numerical solution may be computed accurately and efficiently.

The problem has previously been considered by Drake (1999) who used a boundary element method to solve the integral equation derived from an application of Green's identity. Drake (1999) draws his motivation for studying the problem from the oil industry in which certain oil platforms are supported by a large circular column filled with water and containing a large number of vertical circular pipes. Such oil platforms are known to experience resonant motions (see Drake (1999) for further details) some of which are attributed to the sloshing modes of the water inside the supporting column. The problem under consideration also has relevance to the design of heat exchangers, such as those found in the nuclear industry. The equations that govern the surface elevation in the fluid also apply, under the assumptions of linear acoustic theory, to the pressure field inside a circular guide containing multiple pipes. In the design of heat exchangers involving large arrays of closely-bundled pipes it is of some importance to determine the acoustic resonant frequencies that are exhibited by the configuration and to ensure that they do not coincide with the natural frequencies of any of the individual components.

Advantage is taken of the computational efficiency of the present method to investigate the effect of large evenly distributed arrays of cylinders occupying the tank and to compare these exact results with those from an approximate homogenisation theory also described in Drake (1999). It is shown that the approximation is remarkably accurate, even for tightly packed arrays of cylinders.

2. Formulation and solution

The cylindrical tank and each of the internal cylinders extend uniformly throughout the depth of the fluid, $0 < z < h$, where $z = h$ represents the undisturbed free surface of the fluid and $z = 0$ the bottom of the tank. A cross-section through the tank is sketched for a typical configuration in figure 1. The origin, O, is placed at the centre of the cylindrical tank and from which both Cartesian (x, y) and polar coordinates, (r_0, θ_0) are employed with $x = r_0 \cos\theta_0$, $y = r_0 \sin\theta_0$. The N internal cylinders are centred at (x_j, y_j) and have radius a_j, $j = 1, \ldots, N$. Anticipating later notation we also attribute polar representations of the centres of each of the cylinders with respect to the origin by writing $x_j = R_{0j} \cos\alpha_{0j}$, $y_j = R_{0j} \sin\alpha_{0j}$. In addition to the coordinate system (r_0, θ_0), local polar coordinates (r_j, θ_j) are introduced, based on the centre of the jth internal cylinder. The relative distance and angle between the jth and kth cylinder are defined respectively by

$$R_{jk} = [(x_j - x_k)^2 + (y_j - y_k)^2]^{\frac{1}{2}} \quad \text{and} \quad \alpha_{jk} = \tan^{-1}\left(\frac{y_k - y_j}{x_k - x_j}\right), \tag{1}$$

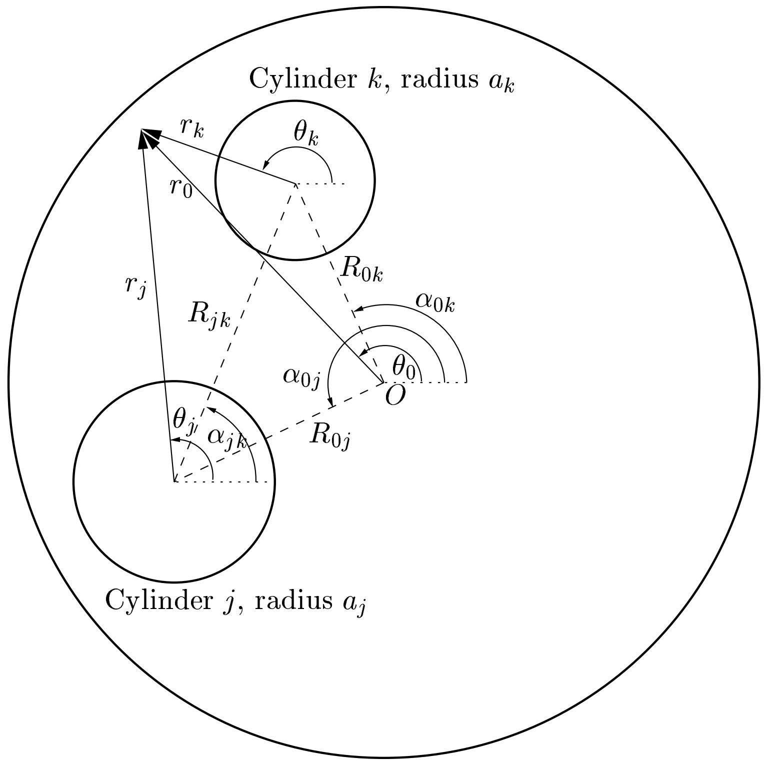

Figure 1: The coordinate system for the circular tank containing multiple cylinders.

for $1 \leq j, k \leq N$. Note that $R_{kj} = R_{jk}$ and $\alpha_{kj} = \alpha_{jk} + \pi$, and this definition may be extended to give meaning to R_{j0} and α_{j0}. See figure 1 for an overview of all these definitions.

Assuming a time-harmonic dependence of angular frequency ω, and variation with the depth coordinate proportional to $\cosh kz$, where k satisfies $\omega^2/g = k \tanh kh$ (g graviational acceleration), the two-dimensional velocity potential, ϕ, satisfies

$$(\nabla^2 + k^2)\phi = 0, \qquad \text{in the fluid domain, } D \tag{2}$$

with the no flow conditions,

$$\frac{\partial \phi}{\partial r_j} = 0, \qquad \text{on } r_j = a_j,\ j = 0, 1, \ldots, N. \tag{3}$$

Clearly, if ϕ satisfies (2) and (3) then so does $\overline{\phi}$, its complex conjugate. Thus we may consider ϕ to be real without loss of generality. However, we prefer to continue with ϕ defined as a complex function as this reduces the mathematical detail. We do so in the knowledge that taking the real part of ϕ at any stage also represents a solution of the problem.

Using the linearity of the governing equations we decompose the total potential into the fundamental potential for the cylindrical tank in the absence of internal cylinders, ϕ_0, plus a sum of 'scattering' potentials, ϕ_j^s associated with each of the N internal cylinders and which describe locally outgoing waves. This is done by writing

$$\phi(x, y) = \phi_0(r_0, \theta_0) + \sum_{j=1}^{N} \phi_j^s(r_j, \theta_j) \tag{4}$$

where

$$\phi_0(r_0, \theta_0) = \sum_{n=-\infty}^{\infty} a_n^{(0)} Z_n^{(0)} J_n(kr_0) \mathrm{e}^{\mathrm{i}n\theta_0}, \quad \text{and} \quad \phi_j^s(r_j, \theta_j) = \sum_{n=-\infty}^{\infty} a_n^{(j)} Z_n^{(j)} H_n(kr_j) \mathrm{e}^{\mathrm{i}n\theta_j} \tag{5}$$

where $J_n(x)$ and $H_n(x) = J_n(x) + \mathrm{i}Y_n(x)$ are the Bessel function and the Hankel function of the first kind, respectively. Also the factors

$$Z_n^{(0)} = H_n'(ka_0)/J_n'(ka_0) \quad \text{and} \quad Z_n^{(j)} = J_n'(ka_j)/H_n'(ka_j), \qquad j = 1, \dots, N$$

are introduced for later convenience. The coefficients $a_n^{(j)}$, $j = 0, 1, \dots, N$ appearing in (5) are to be determined and this will be done by the satisfaction of the cylinder boundary conditions, (3). In order to carry out this task two separate applications of Graf's addition theorem are employed: one is used to satisfy (3) for $j = 1, \dots, N$ and the other for the same boundary condition with $j = 0$ – i.e. on the outer cylinder $r_0 = a_0$.

The system of equations that determine the sloshing modes turn out to be given by

$$a_m^{(k)} + \sum_{n=-\infty}^{\infty} a_n^{(0)} Z_n^{(0)} J_{n-m}(kR_{0k}) \mathrm{e}^{\mathrm{i}(n-m)\alpha_{0k}} + \sum_{\substack{j=1 \\ j \neq k}}^{N} \sum_{n=-\infty}^{\infty} a_n^{(j)} Z_n^{(j)} H_{n-m}(kR_{jk}) \mathrm{e}^{\mathrm{i}(n-m)\alpha_{jk}} = 0 \tag{6}$$

for $m \in \mathbb{Z}$, $k = 1, \dots, N$ and a corresponding equation for $k = 0$,

$$a_m^{(0)} + \sum_{j=1}^{N} \sum_{n=-\infty}^{\infty} a_n^{(j)} Z_n^{(j)} J_{n-m}(kR_{j0}) \mathrm{e}^{\mathrm{i}(n-m)\alpha_{j0}} = 0, \tag{7}$$

for $m \in \mathbb{Z}$.

It is a trivial matter to confirm from the system of equations above that for a single concentric cylinder in the tank, the sloshing modes are determined by the roots of $J_m'(ka_1)Y_m'(ka_0) - J_m'(ka_0)Y_m'(ka_1) = 0$, $m = 0, 1, \dots$ as expected.

In a similar manner to Linton & Evans (1990), we are also able to reintroduce the systems of equations in (6) and (7) into local expressions for the potential to obtain the simplified expressions

$$\phi(r_k, \theta_k) = \sum_{n=-\infty}^{\infty} a_n^{(k)} \left[Z_n^{(k)} H_n(kr_k) - J_n(kr_k) \right] \mathrm{e}^{\mathrm{i}n\theta_k}, \qquad \text{for } r_k < R_{jk}, \forall j \tag{8}$$

and

$$\phi(r_0, \theta_0) = \sum_{n=-\infty}^{\infty} a_n^{(0)} \left[Z_n^{(0)} J_n(kr_0) - H_n(kr_0) \right] \mathrm{e}^{\mathrm{i}n\theta_0}, \qquad \text{for } r_0 > R_{j0}, \forall j \tag{9}$$

3. Results

The solution of the sloshing problem is determined by the non-trivial solutions of the homogeneous system of equations given by (6) and (7) combined. Numerical solutions of these equations are found by truncating the infinite sums to sums between $-M$ and M. Numerical experimentation determines appropriate values of M for a specified accuracy, and $M = 5$ was found to be sufficient for five decimal places accuracy in most cases. The frequency of sloshing is determined in terms of the non-dimensional parameter ka_0.

Of particular interest is the effect on sloshing frequencies due to a large number of evenly distributed cylinders. The method presented here is capable of efficiently computing accurate results for N of the order of 100. However, for such large numbers of cylinders it is reasonable to appeal to homogenisation techniques to argue that the effect of the large number of discrete cylinders is approximately equivalent to a new homogeneous medium with an altered wavenumber.

Just such an approximation appears in Drake (1999) for a multitude of evenly spaced cylinders of 'small' cross-section and results in the simple relation

$$k^2 \approx (1-\sigma)k_0^2, \quad \text{where} \qquad J_n'(k_0 a_0) = 0, \quad n = 0, 1, \ldots \tag{10}$$

and σ (assumed small) is the proportion of the cross-sectional area of the tank occupied by the cylinders. It can be seen from (10) that the wavenumber in a medium occupied by many cylinders is reduced by a factor of $(1-\sigma)^{1/2}$ from the wavenumber for sloshing tank in the absence of internal cylinders.

In figure 2 curves using the homogenisation theory defined by (10) are plotted with values of ka_0 computed for arrays of $N = 24, 44$ and 96 cylinders and show remarkable agreement even for large cylinders. The corresponding modal amplitudes for two extreme cases appearing in figure 2 are shown in figure 3.

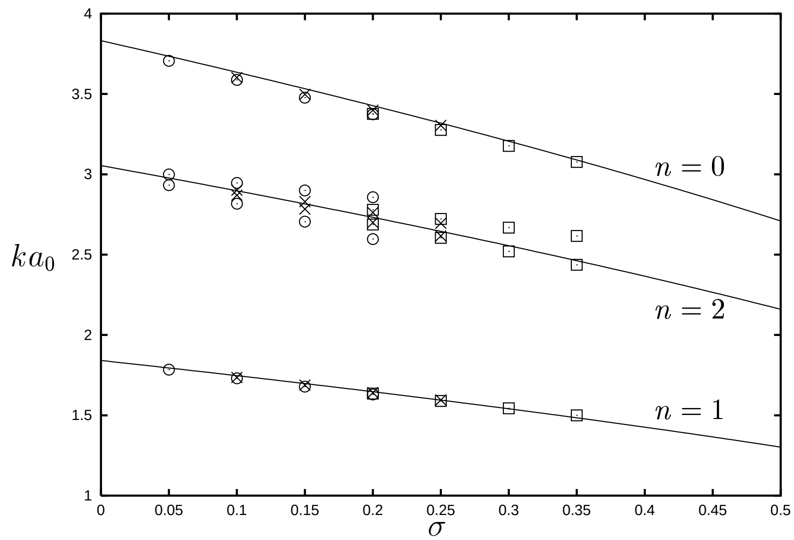

Figure 2: Variation of ka_0 for sloshing against σ predicted by the approximation (10). Mode numbers are labelled against curves. Data computed for rectangular arrays of cylinders are shown in symbols for $N = 24$ (circles), $N = 44$ (crosses) and $N = 96$ (boxes).

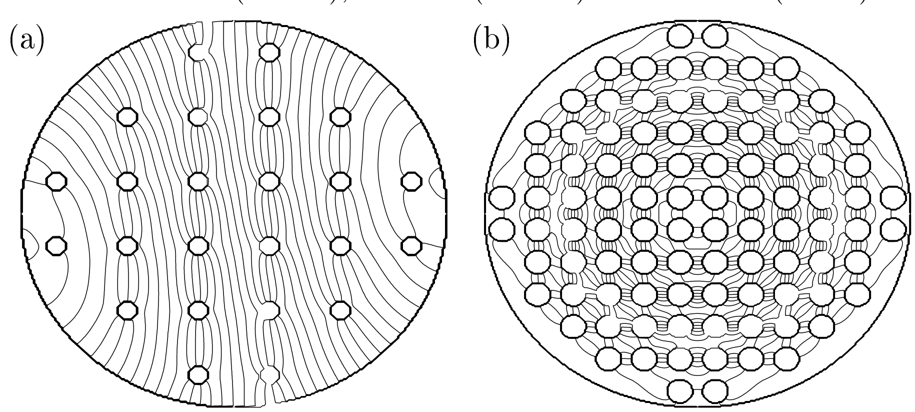

Figure 3: Contour plots of the amplitude of the sloshing modes: (a) $N = 24$, $\sigma = 0.05$, mode $n = 1$, $ka_0 = 1.78445$; (b) $N = 96$, $\sigma = 0.35$, mode $n = 0$, $ka_0 = 3.07707$.

4. References

1. Drake, K. S. 1999 The effect of internal pipes on the fundamental frequency of liquid sloshing in a circular tank. *Appl. Ocean. Res.* **21**, 133–143.

2. Linton, C. M. & Evans, D. V. 1990 The interaction of waves with arrays of vertical circular cylinders. *J. Fluid Mech.* **215**, 549–569.