Embedding formulas for scattering by an arbitrary configuration of parallel breakwaters

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SUMMARY

In [1] it was noted that the solution to the scattering of a plane wave by a finite gap in an infinite, thin breakwater for arbitrary incident wave angle could be written in terms of the solution to the problem in which the incident wave is parallel to the breakwater. Such a relationship is an example of an embedding formula. Recently it has been shown [2] that embedding formulas can be derived for the case of N gaps in an infinite breakwater; in this case the result for an arbitrary angle of incidence is determined from the solution to $2N$ separate problems (the reason this reduces to 1 in the case of a single gap is due to symmetry).

Solutions to these breakwater-gap problems are related to problems in which the gaps and breakwaters are interchanged. (This is an example of Babinet's principle, see [3].) Thus embedding formulas are easily derived for a finite number of collinear breakwaters. Here we show that similar formulas can be derived when the breakwaters are parallel, but not necessarily collinear; a situation which does not correspond to any breakwater-gap problem.

FORMULATION

We consider an array of N breakwaters, each in the form of a thin vertical barrier extending throughout the water depth. The breakwaters are all parallel so that their intersection with the undisturbed free surface, denoted by L, is the union of a finite collection of parallel strips in $\mathbb{R}^2$:

$$L = \bigcup_{n=1}^{N} L_n, \qquad L_n = \{\mathbf{x} : a_n < x < b_n,\, y = \eta_n\},$$

where $\mathbf{x} = (x, y)$. A plane wave making an angle β with the x-axis is of the form $\mathrm{Re}\{f_\beta(\mathbf{x}) \cosh k(z+h) \exp(-\mathrm{i}\omega t)\}$, where h is the water depth, $k \tanh kh = \omega^2/g$, and

$$f_\beta(\mathbf{x}) = \mathrm{e}^{-\mathrm{i}k(x\cos\beta + y\sin\beta)}.$$

The total scattered field will be characterized by $f_\beta + \phi_\beta$, where

$$(\nabla^2 + k^2)\phi_\beta = 0 \qquad \mathbf{x} \in \mathbb{R}^2 \backslash L, \tag{1}$$

$$\frac{\partial \phi_\beta}{\partial y} = \mathrm{i}k \sin\beta\, f_\beta \qquad \mathbf{x} \in L. \tag{2}$$

The potential ϕ_β also satisfies an outgoing radiation condition as $r = \sqrt{x^2 + y^2} \to \infty$, and appropriate conditions at the edges of the breakwaters.

We will also consider $N + 1$ further (non-physical) problems where instead of the body boundary condition (2), we have

$$\psi_\beta = -f_\beta \qquad \mathbf{x} \in L$$

and

$$\psi_\beta^{(n)} = -f_\beta^{(n)} \qquad \mathbf{x} \in L, \qquad n = 1, \ldots, N,$$

where

$$f_\beta^{(n)}(\mathbf{x}) = \begin{cases} f_\beta(\mathbf{x}) & \mathbf{x} \in L_n, \\ 0 & \mathbf{x} \in L \backslash L_n. \end{cases}$$

Note that $\psi_\beta = \sum_{n=1}^{N} \psi_\beta^{(n)}$.

The solutions to these problems are related. Consider the function

$$\phi_\beta = \frac{\mathrm{i}}{k\sin\beta}\left(\frac{\partial \psi_\beta}{\partial y} + \sum_{n=1}^{N}\left[A_{-,\beta}^{(n)}\frac{\partial \psi_0^{(n)}}{\partial y} + A_{+,\beta}^{(n)}\frac{\partial \psi_\pi^{(n)}}{\partial y}\right]\right) \qquad \beta \neq 0, \pi. \tag{3}$$

By construction ϕ_β satisfies (1), (2) and the radiation condition. The constants $A^{(n)}_{\pm,\beta}$ are to be chosen so that ϕ_β is bounded at (a_n,η_n), (b_n,η_n), $n=1,\dots N$. The number of unknown constants is the same as the number of breakwater edges and this provides the reason why the embedding formulas given below require the solution of $2N$ separate problems.

All the above problems can easily be formulated as integral equations. If we define

$$\mathbf{x}'=(\xi,\eta),\quad R=|\mathbf{x}-\mathbf{x}'|,$$

then the breakwater scattering problem can be written as an integral equation for the unknown jump in the potential across each barrier. For numerical computations one would probably formulate the problem as a hypersingular integral equation, but for our purposes we use the fact that all the barriers are parallel to the x-axis to write the integral equation in the form

$$\left(\frac{\partial^2}{\partial x^2}+k^2\right)\int_L p_\beta(\mathbf{x}')H_0^{(1)}(kR)\,\mathrm{d}\xi=2k\sin\beta\, f_\beta(\mathbf{x}),\quad \mathbf{x}\in L, \tag{4}$$

where

$$p_\beta(x,\eta_n)\equiv\tfrac{1}{2}(\phi_\beta(x,\eta_n^+)-\phi_\beta(x,\eta_n^-)).$$

Note that $p_\beta(a_n,\eta_n)=p_\beta(b_n,\eta_n)=0$. Once p_β has been determined, the solution is given everywhere in the fluid by

$$\phi_\beta(\mathbf{x})=-\frac{\mathrm{i}}{2}\frac{\partial}{\partial y}\int_L p_\beta(\mathbf{x}')H_0^{(1)}(kR)\,\mathrm{d}\xi. \tag{5}$$

For ψ_β and $\psi_\beta^{(n)}$ we have

$$\int_L v_\beta(\mathbf{x}')H_0^{(1)}(kR)\,\mathrm{d}\xi=-2\mathrm{i}f_\beta(\mathbf{x}),\quad \mathbf{x}\in L, \tag{6}$$

where $v_\beta(x,\eta_n)\equiv\frac{1}{2}(\partial\psi_\beta/\partial y|_{y=\eta_n^+}-\partial\psi_\beta/\partial y|_{y=\eta_n^-})$, and

$$\int_L v_\beta^{(n)}(\mathbf{x}')H_0^{(1)}(kR)\,\mathrm{d}\xi=-2\mathrm{i}f_\beta^{(n)}(\mathbf{x}),\qquad \mathbf{x}\in L, \tag{7}$$

where $v_\beta^{(n)}(x,\eta_n)\equiv\frac{1}{2}(\partial\psi_\beta^{(n)}/\partial y|_{y=\eta_n^+}-\partial\psi_\beta^{(n)}/\partial y|_{y=\eta_n^-})$.

As $kr\to\infty$ ($x=r\cos\theta$, $y=r\sin\theta$), we have

$$\psi_\beta\sim\frac{\mathrm{e}^{\mathrm{i}(kr-3\pi/4)}}{(2\pi kr)^{1/2}}G_{\theta,\beta},\qquad \phi_\beta\sim\frac{\mathrm{e}^{\mathrm{i}(kr-3\pi/4)}}{(2\pi kr)^{1/2}}F_{\theta,\beta},$$

where the diffraction coefficients $G_{\theta,\beta}$ and $F_{\theta,\beta}$ are given by

$$G_{\theta,\beta}=\int_L f_\theta v_\beta,\qquad F_{\theta,\beta}=\mathrm{i}k\sin\theta\int_L f_\theta p_\beta. \tag{8}$$

It follows from (4) and (6) that $G_{\theta,\beta}=G_{\beta,\theta}$ and $F_{\theta,\beta}=F_{\beta,\theta}$.

The particular form of the integral equation (4) is used because it can be written

$$\left(\frac{\partial}{\partial x}\pm\mathrm{i}k\right)\left(\frac{\partial}{\partial x}\mp\mathrm{i}k\right)\int_L p_\beta(\mathbf{x}')H_0^{(1)}(kR)\,\mathrm{d}\xi=2k\sin\beta\, f_\beta(\mathbf{x}),\qquad \mathbf{x}\in L,$$

from which, using integration by parts,

$$\left(\frac{\partial}{\partial x}\pm\mathrm{i}k\right)\int_L\left(\frac{\partial p_\beta}{\partial\xi}(\mathbf{x}')\mp\mathrm{i}kp_\beta(\mathbf{x}')\right)H_0^{(1)}(kR)\,\mathrm{d}\xi=2k\sin\beta\, f_\beta(\mathbf{x}),\qquad \mathbf{x}\in L.$$

Now we solve this pair of first-order ODEs (each defined on N intervals):

$$\int_L\left(\frac{\partial p_\beta}{\partial\xi}(\mathbf{x}')\mp\mathrm{i}kp_\beta(\mathbf{x}')\right)H_0^{(1)}(kR)\,\mathrm{d}\xi=\frac{2\mathrm{i}\sin\beta}{\cos\beta\mp1}\left(f_\beta(\mathbf{x})+\sum_{n=1}^{N}c^{(n)}_{\mp,\beta}f^{(n)}_{\mp}(\mathbf{x})\right),\qquad \mathbf{x}\in L, \tag{9}$$

where, for convenience, we use subscripts $+$ and $-$ for π and 0, respectively, and $c^{(n)}_{\mp,\beta}$ are $2N$ constants of integration.

It follows from (6) and (7) that

$$\frac{\partial p_\beta}{\partial x} \mp \mathrm{i}kp_\beta = \frac{-\sin\beta}{\cos\beta \mp 1}\left(v_\beta + \sum_{n=1}^{N} c^{(n)}_{\mp,\beta} v^{(n)}_{\mp}\right), \qquad \mathbf{x} \in L, \tag{10}$$

and hence (multiply by $f^{(m)}_{\mp}(\mathbf{x})$ and integrate over L)

$$G^{(m)}_{\mp,\beta} + \sum_{n=1}^{N} c^{(n)}_{\mp,\beta} G^{(m,n)}_{\mp,\mp} = 0, \qquad m = 1, \ldots, N, \tag{11}$$

where

$$G^{(m,n)}_{\theta,\beta} = \int_L f^{(m)}_\theta v^{(n)}_\beta \, \mathrm{d}\xi = G^{(n,m)}_{\beta,\theta}, \tag{12}$$

and

$$G^{(m)}_{\theta,\beta} = \int_L f^{(m)}_\theta v_\beta \, \mathrm{d}\xi = \int_L f_\beta v^{(m)}_\theta \, \mathrm{d}\xi = \sum_{n=1}^{N} G^{(n,m)}_{\beta,\theta} = \sum_{n=1}^{N} G^{(m,n)}_{\theta,\beta}.$$

Equation (11) represents two $N \times N$ systems of equations for the constants $c^{(n)}_{\mp,\beta}$. Note that knowledge of $v^{(n)}_{\pm}$ is sufficient to be able to compute these constants.

The constants $A^{(n)}_{\pm,\beta}$ defined in (3) and $c^{(n)}_{\mp,\beta}$ defined in (9) can be shown to be related. Using (3), (6) and (7) we have

$$\int_L p_\beta(\mathbf{x}') H^{(1)}_0(kR) \, \mathrm{d}\xi = \frac{2}{k\sin\beta}\left(f_\beta(\mathbf{x}) + \sum_{n=1}^{N}\left[A^{(n)}_{-,\beta} f^{(n)}_0(\mathbf{x}) + A^{(n)}_{+,\beta} f^{(n)}_\pi(\mathbf{x})\right]\right), \tag{13}$$

for $\mathbf{x} \in L$. If we differentiate with respect to x, use integration by parts, and then use (10) we can show that

$$A^{(n)}_{\pm,\beta} = \tfrac{1}{2}(1 \mp \cos\beta) c^{(n)}_{\pm,\beta}. \tag{14}$$

We also get (multiply (13) by $v_\theta(\mathbf{x})$ and integrate over L)

$$F_{\theta,\beta} = -\frac{\sin\theta}{\sin\beta}\left(G_{\beta,\theta} + \sum_{n=1}^{N}\left[A^{(n)}_{-,\beta} G^{(n)}_{0,\theta} + A^{(n)}_{+,\beta} G^{(n)}_{\pi,\theta}\right]\right), \qquad \beta \neq 0, \pi. \tag{15}$$

If we eliminate v_β from (10) we obtain

$$\frac{\partial p_\beta}{\partial x} + \mathrm{i}k\cos\beta \, p_\beta = \tfrac{1}{2}\sin\beta \sum_{n=1}^{N}\left(c^{(n)}_{-,\beta} v^{(n)}_0 - c^{(n)}_{+,\beta} v^{(n)}_\pi\right), \qquad \mathbf{x} \in L, \tag{16}$$

and this can be solved for p_β. For $\mathbf{x} \in L_m$ we thus have

$$p_\beta(x, \eta_m) = \tfrac{1}{2} f_\beta(x, \eta_m) \sin\beta \int_{a_m}^{x} \sum_{n=1}^{N} f_{\pi-\beta}(\xi, \eta_n)\left(c^{(n)}_{-,\beta} v^{(n)}_0(\xi, \eta_n) - c^{(n)}_{+,\beta} v^{(n)}_\pi(\xi, \eta_n)\right) \mathrm{d}\xi, \tag{17}$$

which expresses p_β in terms of $v^{(n)}_0$ and $v^{(n)}_\pi$. The solution to the breakwater scattering problem for an arbitrary angle of incidence is thus determined, through (5), in terms of the solution to $2N$ integral equations of the form (7).

The relationships between the diffraction coefficients take particularly simple forms if we define

$$\mathcal{F}_{\theta,\beta} = (\cos\theta + \cos\beta) F_{\theta,\beta}, \qquad \mathcal{G}_{\theta,\beta} = (\cos\theta + \cos\beta) G_{\theta,\beta}, \quad \text{etc.}$$

From (8) and (17), we can derive

$$\mathcal{F}_{\theta,\beta} = \tfrac{1}{2}\sin\theta\sin\beta \sum_{n=1}^{N}\left(c^{(n)}_{-,\beta} G^{(n)}_{0,\theta} - c^{(n)}_{+,\beta} G^{(n)}_{\pi,\theta}\right)$$

and from (15) we then get

$$\mathcal{G}_{\theta,\beta} = -\sum_{n=1}^{N}\left(A^{(n)}_{-,\beta} \mathcal{G}^{(n)}_{0,\theta} + A^{(n)}_{+,\beta} \mathcal{G}^{(n)}_{\pi,\theta}\right). \tag{18}$$

Alternatively, using (11),

$$\mathcal{F}_{\theta,\beta} = \tfrac{1}{2}\sin\theta\sin\beta\sum_{n=1}^{N}\sum_{m=1}^{N}\left(c_{+,\beta}^{(n)}c_{+,\theta}^{(m)}G_{\pi,\pi}^{(n,m)} - c_{-,\beta}^{(n)}c_{-,\theta}^{(m)}G_{0,0}^{(n,m)}\right)$$

and

$$\mathcal{G}_{\theta,\beta} = \sum_{n=1}^{N}\sum_{m=1}^{N}\left(A_{-,\beta}^{(n)}A_{-,\theta}^{(m)}\mathcal{G}_{0,0}^{(n,m)} + A_{+,\beta}^{(n)}A_{+,\theta}^{(m)}\mathcal{G}_{\pi,\pi}^{(n,m)}\right),$$

from which the symmetry relations $F_{\beta,\theta} = F_{\theta,\beta}$ and $G_{\beta,\theta} = G_{\theta,\beta}$ are obvious. In fact, equation (18) follows from (15) if we impose the symmetry of F.

CONCLUSION

We have shown that the scattering problem for a plane wave incident from an arbitrary angle on an arbitrary configuration of N parallel breakwaters extending throughout the water depth can be related to $2N$ separate scattering problems for the same geometry, but with different boundary conditions. For a given geometrical configuration and frequency, we must solve $2N$ integral equations (each with the same logarithmically singular kernel) and invert two $N \times N$ systems of equations in order to determine the solution for any incident wave angle. This can lead to significant computational savings if solutions are required over a range of angles.

Embedding formulas can also be derived for the case of N gaps in a breakwater which has finite thickness, see [4]. In this situation we require the solution to $4N$ problems, corresponding to the fact that there are $4N$ corners. Although Babinet's principle does not apply here, it appears that it may be possible to derive embedding formulas for an array of rectangular columns, and work is currently underway to establish whether this is indeed the case and, if it is, what form such formulas would take.

References

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