EXACT SOLUTIONS OF FLOATING ELASTIC PLATE PROBLEM

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SUMMARY
An inverse method for solving the hydroelastic problem of floating plate is introduced. The method provides exact solutions which are suggested to use for testing known numerical algorithms. It was found that the numerical method presented at the previous Workshop accurately predicts the hydrodynamic loads but fails to describe the plate deflections for high frequencies of external excitation. The developed procedure can be recommended for testing other numerical methods.

1. INTRODUCTION

Very large floating structures (VLFS) are considered at present as alternatives to huge land-based facilities such as airports. A possible configuration of the structures is an elastic thin plate of large horizontal extends. Hydroelastic response of the floating plate in regular waves is a challenging topic which received a particular attend during last years. In order to predict accurately the behaviour of a floating elastic plate in waves, several numerical algorithms have been developed [1-5]. The obtained numerical results were compared both between each other and with available experimental data. Also experiments with the numerical algorithms were performed to study their convergence and accuracy of the results. This great interest to numerical algorithms is explained by the huge dimensions of the VFLS, when sea trials are very expensive and laboratory experiments with small models are questionable. It is well recognized that the numerical modeling of the floating structure behaviour is the main tool for the design purposes. At the initial design stage simple models and numerical schemes can be used, in order to estimate amplitudes of the both plate deflections and the elastic stresses. However, at the following stages more accurate models and computer codes must be employed.

Generally speaking, it is not easy to decide which numerical algorithm is the best one. Comparison of numerical results obtained by different methods usually does not allow us to arrive at a final decision. The point is that a tested numerical method has to provide the solution of an original mathematical model at a prescribed accuracy. It is rather difficult to test a numerical method with the help of another numerical method.

In order to derive an adequate numerical algorithm for solving the hydroelastic problem, one needs to study first the corresponding mathematical model and features of its solution. For the problem of floating elastic plate such a study has not been performed yet. This can be well explained by the complexibility of the hydroelastic problem.

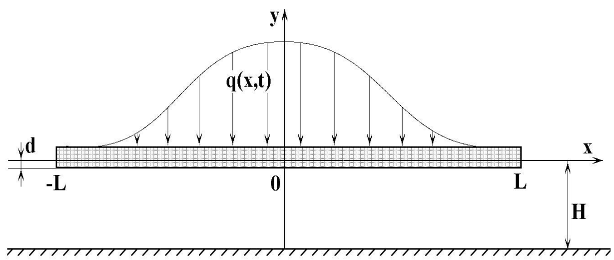
The problem is linear, but coupled: hydrodynamics and structural dynamics problems have to be solved simultaneously. Difficulties are connected with the floating plate response to high-frequency excitation, that is to short incoming waves.

Main attention in numerical studies of the VLFS is focused on periodic in time behaviour of the structure in regular waves. Without accurate solution of this problem one cannot expect to get reliable results in other practical problems such as, for example, floating plate response to external loads. This statement looks correct and obvious. However, it can be shown that the latter problem can be solved by an inverse method in contrast to the simpler problem of floating plate behaviour in regular waves. The corresponding solutions can be obtained with a prescribed accuracy and can be used to test numerical algorithms.

The inverse method is used in the present analysis of hydroelastic behaviour of a floating plate under an external and periodical in time load. Within the inverse method the distribution of the hydrodynamic pressures under the floating plate is assumed given and shape of the liquid boundary is evaluated solving the hydrodynamic part of the original problem. Identifying the plate deflection with the liquid surface shape in the region, where the hydrodynamic pressures are different from the atmospheric pressure, and using the plate equation, one can reconstruct the distribution of the external loads along the plate. It should be noted that the distribution of the hydrodynamic pressures cannot be arbitrary due to the boundary conditions on the plate edge. These conditions can be readily satisfied in the two-dimension problem, which is used below for illustration of the inverse method and obtained exact solutions.

2. FORMULATION OF THE PROBLEM

The plane unsteady problem of hydroelastic behaviour of a plate floating on a liquid surface is considered within the framework of linear theory. The plate vibration is caused by periodic external load of frequency ω and small amplitude B. The plate deflection is governed by the Euler beam equation. The end points of the beam are free of stresses. Both the plate thickness h and plate draft at rest d are much smaller than the plate length $2L$ and the liquid depth H. The external load $Bq(x', t'\omega)$, where $|x'| < L$ and $|q(x't'\omega)| \leq 1$, is symmetric with respect to the plate centre. The liquid is ideal and incompressible, its flow is plane, potential and symmetric.



Non-dimensional variable are used below. The half-length of the beam L is taken as the length scale; $1/\omega$ as the time scale; the quantity B as the pressure scale; $B/(\rho g)$ as the deflection scale, where ρ is the liquid density and g is the acceleration due to gravity; $2LdB$ as the scale of bending moments, and $\omega BL/(\rho g)$ as the scale of the velocity potential.

Within the linear theory the non-dimensional velocity potential of the flow $\phi(x, y, t)$ satisfies the following equations

$$\phi_{xx} + \phi_{yy} = 0 \quad (-\infty < x < +\infty, \ -H_0 < y < 0),$$

$$\phi_y = 0 \quad (y = -H_0),$$

$$\phi_y = \eta_t, \qquad \gamma\phi_t + \eta = 0 \qquad (y = 0, \ |x| > 1),$$

$$\phi_y = w_t(x, t), \qquad (y = 0, \ |x| < 1),$$

where $H_0 = H/L$ and $\gamma = L\omega^2/g$, equation $y = \eta(x, t)$ describes the free surface shape. The deflection $w(x, t)$ of the plate is governed in the non-dimensional variables by the Euler beam equation

$$\alpha w_{tt} + \beta w_{xxxx} = p(x, t) - q(x, t) \quad (|x| < 1),$$

$$w_{xx}(\pm 1, t) = 0, \qquad w_{xxx}(\pm 1, t) = 0,$$

where $p(x, t)$ is the hydrodynamic pressure along the beam, which is given by the linearized Bernulli's equation

$$p(x, t) = -\gamma\phi_t(x, 0, t) - w(x, t),$$

the function $q(x, t)$ describes the distribution of the external loads on the beam; $\alpha = \gamma d/L, \quad \beta = EJ/(\rho g L^4)$ are non-dimensional parameters of the problem. Here E is the elasticity modulus and J is the inertia momentum of the beam cross-section, $J = h^3/12$ for the beam of uniform thickness h.

In the periodic case the problem solution is sought in the form

$$\phi(x, y, t) = \mathrm{Re}\,[i \exp{(it)} \Phi(x, y)],$$

$$w(x, t) = \mathrm{Re}\,[\exp{(it)} W(x)],$$

$$p(x, t) = \mathrm{Re}\,[\exp{(it)} P(x)],$$

$$q(x, t) = \mathrm{Re}\,[\exp{(it)} Q(x)].$$

The new unknown complex-valued functions $\Phi(x, y)$, $W(x)$ and $P(x)$ satisfy the following equations and the boundary conditions:

$$\Phi_{xx} + \Phi_{yy} = 0 \quad (-\infty < x < +\infty, \ -H_0 < y < 0), \tag{1}$$

$$\Phi_y = 0 \quad (y = -H_0), \tag{2}$$

$$\Phi_y = \gamma\Phi \quad (y = 0, \ |x| > 1), \tag{3}$$

$$\Phi_y = W(x) \quad (y = 0, \ |x| < 1), \tag{4}$$

$$P(x) = \gamma\Phi(x, 0) - W(x) \qquad (|x| < 1), \tag{5}$$

$$\beta\, W^{IV} - \alpha\, W \ = \ P(x) - Q(x) \qquad (|x| < 1), \tag{6}$$

$$W''(\pm 1) = 0, \qquad W'''(\pm 1) = 0. \tag{7}$$

The velocity potential $\Phi(x, y)$ satisfies also the radiation condition as $|x| \to \infty$.

Equations (1)-(5) represent the hydrodynamic part of the original coupled problem and lead to the integral equation

$$P(x) + \frac{\gamma}{2\pi} \int_{-1}^{1} P(x_0) K(x - x_0) dx_0 = -W(x) \tag{8}$$

with respect to the hydrodynamic pressure $P(x)$ along the plate. The function $K(z)$ is given as

$$K(z) = -2\pi i \frac{k e^{-ik|z|}}{H_0(k^2 - \gamma^2) + \gamma} + 2\pi \sum_{j=1}^{\infty} \frac{s_j e^{-s_j|z|}}{H_0(s_j^2 + \gamma^2) - \gamma},$$

where k is the non-dimensional wavenumber that is a positive solution of the equation $k \tanh kH_0 = \gamma$, $s_j = (\pi j - \delta_j)/H_0$ and δ_j is the solution of the equation $\delta_j = \arctan(\gamma H_0/(\pi j - \delta_j))$, $j \geq 1$.

We shall determine the functions $W(x)$ and $P(x)$, which satisfy the integral equation (8), the differential equation (6) and the boundary conditions (7) for a given functions $Q(x)$.

The problem is complicated and its solution can be obtained only numerically. Below an inverse method is used to derive exact solutions of the problem.

3. INVERSE METHOD

The inverse method is based on the following idea. Let's assume that hydrodynamic pressure $P(x)$ is given. Then the plate deflection $W(x)$ can be calculated using (8) and the corresponding external load $Q(x)$ can be evaluated from equation (6). It should be noted that this procedure is straightforward and the function $Q(x)$ can be obtained with a prescribed accuracy. Within the inverse problem the function $P(x)$ is given and both the functions $Q(x)$ and $W(x)$ are unknown.

The given function $P(x)$ cannot be arbitrary and has to satisfy some restrictions. In general symmetric problem,

$$P(1) = P(-1) \neq 0$$

$$P(x) - P(1) = O((1 - x)\ln(1 - x)) \qquad (1 - x) \to +0.$$

Therefore, in order to calculate $W''(\pm 1)$ and $W'''(\pm 1)$ and to check the boundary conditions (7), we cannot simply differentiate both sides of equation (8). This is why we considered here the case, where

$$P(1) = P'(1) = P''(1) = P'''(1) = 0 \qquad (9)$$

and equation (8) can be differentiated four times with respect to x, where $-1 \leq x \leq 1$. It can be shown that equalities (9) imply continuous behaviour of the liquid surface at the plate edges.

Consider real functions $f_j(x)$, $j = 1, 2, 3$, which satisfy conditions (9) and are even. The hydrodynamic pressure is written as

$$P(x) = K_1 f_1(x) + K_2 f_2(x) + K_3 f_3(x), \qquad (10)$$

where complex-valued coefficients K_1, K_2 and K_3 are determined with the help of the boundary conditions (7) and the scaling rule $max|Q(x)| = 1$. Substituting (10) into (8), we obtain

$$W(x) = K_1 W_1(x) + K_2 W_2(x) + K_3 W_3(x), \qquad (11)$$

where $W_j(x)$ are complex-valued, smooth and even functions given by (8) with $P(x)$ being substituted by $f_j(x)$.

The functions $f_j(x)$ are convenient to choose such that the integral term in (8) can be evaluated analytically. In this case accuracy of the calculations within the inverse method can be controlled.

Representation (11) and conditions (7) yield two equations for the coefficients K_j. Substitution of expansions (10) and (11) into the beam equation (6) provides

$$Q(x) = K_1 Q_1(x) + K_2 Q_2(x) + K_3 Q_3(x). \qquad (12)$$

The third equation for the coefficients K_j follows from the equality $max|Q(x)| = 1$. Once the coefficients in (10) have been determined equations (11) and (12) yields exact solution of the inverse problem.

In order to study accuracy of a numerical algorithm, we return to the direct hydroelastic problem (6)-(8), where $Q(x)$ is given by the solution of the inverse problem. The numerical solution of the direct problem has to be compared thereafter with that given by formulae (10) and (11). Solving the inverse problem and then the direct one for different values of the parameter γ and different functions $f_j(x)$, $j = 1, 2, 3$, one can discover peculiarities of the numerical algorithm in use.

4. NUMERICAL RESULTS

Numerical calculations were performed for the conditions of the experiments carried out by Wu *et al* [3] for homogeneous narrow plate in a channel: d =8.36mm, H =1.1m, h =38mm, EJ =471kg m^3/s^2, L =5m. For this characteristics of the plate $\beta = 7.7 \cdot 10^{-5}$. Values of the others parameters presented in the table.

	$\omega(s^{-1})$	$T(s)$	α	γ	k
1	2.2	2.875	0.004	2.43	3.654
2	4.4	1.429	0.016	9.85	10.1
3	8.98	0.7	0.069	41.06	41.06
4	15.7	0.4	0.21	125.8	125.8

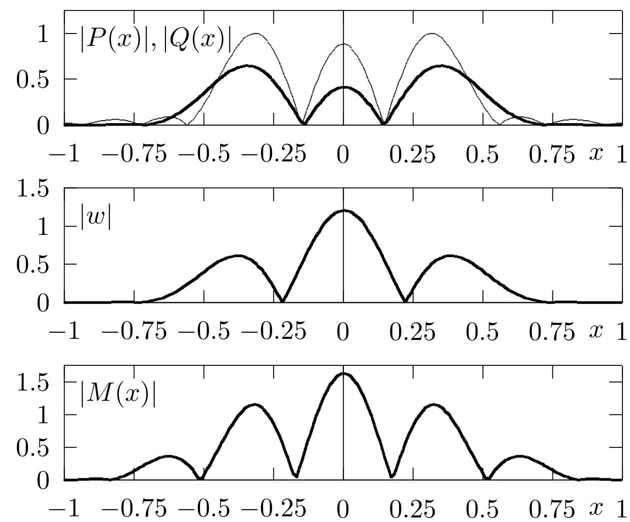


Figure 1: $T = 2.875, \omega = 2.19$

These parameters were used in [4] for the problem of floating elastic plate behaviour in regular waves with $2\pi L/k$ being the length of incident wave. The numerical method presented in [4] is based on expansions of the hydrodynamic pressure and the beam deflection with respect to different basic functions. This made it possible to simplify the treatment of the hydrodynamic part of the problem and at the same time to satisfy accurately the beam boundary conditions.

The numerical results for the incident wave problem were compared with those obtained in [3-5] by different methods. It is shown that the plate deflections and the bending stresses, obtained by the three different methods, are almost identical for low frequencies ($\omega = 4.4s^{-1}$, $\omega = 2.2s^{-1}$) of incident waves. However, for higher frequencies ($\omega = 8.98s^{-1}$) the three methods give quite different amplitudes of both the deflections and the stresses. It is still not clear which method is better and what is a reason for this discrepancy.

The method presented at the previous Workshop [4] was developed further to account for the given external loads on the floating plate. First the inverse method was used with

$$f_1(x) = \cos^4(\frac{\pi x}{2}), \quad f_2(x) = \cos^6(\frac{\pi x}{2}),$$

$$f_3(x) = \cos^8(\frac{\pi x}{2})$$

in (10), to generate the distribution of the external loads $Q(x)$ according to formula (12). Then the direct problem has been solved with the external load distribution described by the function $Q(x)$. For each frequency ω we obtain two curves for the amplitude of the hydrodynamic pressure $|P(x)|$: one of which follows from equation (10) and another one is obtained by the direct method. Both curves are essentially the same for all tested frequencies (see figures 1-4). In

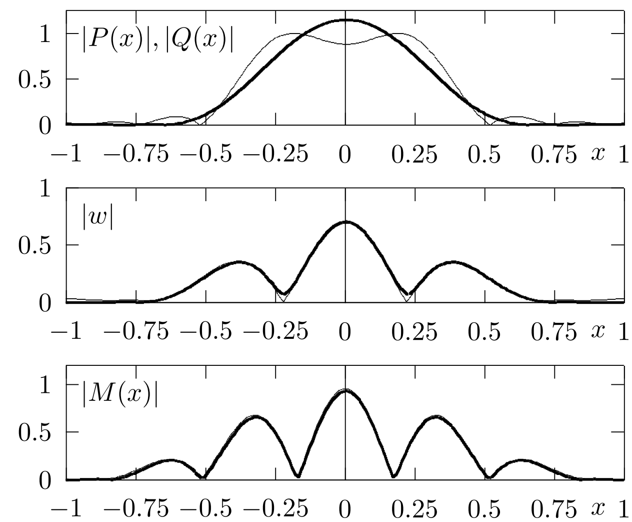


Figure 2: $T = 1.429, \omega = 4.4$

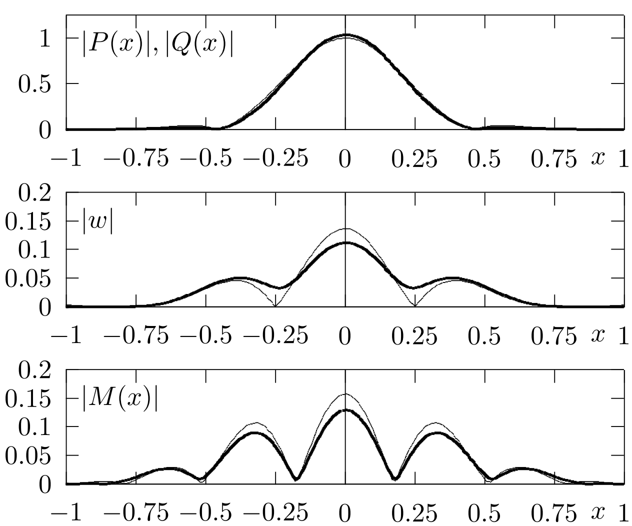


Figure 3: $T = 0.7, \omega = 8.97$

the figures the amplitude of the hydrodynamic pressure are shown with thick solid lines and the amplitude of the external load $|Q(x)|$ with thin solid lines. Correspondingly, there are two curves for both the amplitude of the plate deflection $|W(x)|$ and that of the bending moments $|M(x)|$. The amplitudes predicted by the direct method are shown with thin solid lines and those given by the inverse method with thick solid lines. It is seen that the results are close to each other for low frequencies of excitation (see figures 1 and 2) and the discrepancy between them grow with increasing of the excitation frequency (see figures 3 and 4). Note different scales for the beam deflections and bending moments in the figures.

We may conclude that the hydrodynamic characteristic, which is the pressure distribution, is predicted quite accurately by the method described in [4]. However, the accuracy of computing the elastic character-

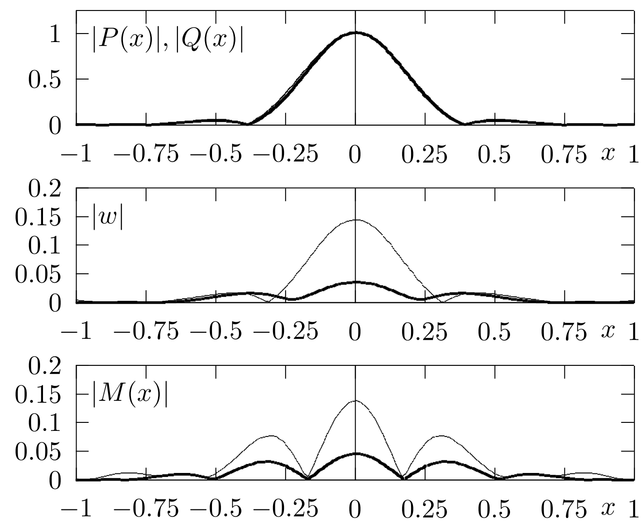


Figure 4: $T = 0.4, \omega = 15.7$

istics, which are the plate deflections and bending moments, drops with the excitation frequency increasing. We expect that within the numerical algorithm presented in [4] the error of numerical results is connected with the treatment of the beam equation (6) but not with numerical treatment of the integral equation (8). The same procedure can be recommended for testing other numerical methods.

5. ACKNOWLEDGMENTS

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Discussion Sheet

Abstract Title :	Exact Solutions of floating elastic plate problem
(Or) Proceedings Paper No. : 20	***Page :*** 077
First Author :	Khabakhpasheva, T.I. and Korobkin, A.A.
Discusser :	Maureen McIver

Questions / Comments :

Is the inverse problem you discuss ill-posed?

Author's Reply :
(If Available)

We call our problem the "inverse problem" because the given and unknown functions change places. In the direct problem the external load Q(x) is a given function and both hydrodynamic pressures P(x) and plate deflection W(x) are unknown. In the framework of the inverse problem, P(x) is a given function and Q(x), W(x) are unknown.
When we consider the inverse problem, we do not solve any equation. We only substitute functions into the formulae, and as a result we obtain exact dependencies.