HYDRODYNAMIC LOADS AT THE EARLY STAGE OF A FLOATING WEDGE IMPACT

A. Iafrati, INSEAN (Italian Ship Model Basin),Italy, E-mail: a.iafrati@insean.it
A. A. Korobkin, Lavrentyev Institute of Hydrodynamics, Novosibirsk, Russia

SUMMARY
The hydrodynamic loads induced in the early stage of an impulsive vertical motion of an initially floating wedge are investigated with the help of a small time expansion procedure. It is found that, for bodies having pronounced flares, flow details about the intersection between the solid boundary and the free surface significantly affect the pressure distribution all along the wetted part of the body and then the total hydrodynamic force.

1. INTRODUCTION

The hydrodynamic loads experienced by an initially floating body during its impulsive vertical motion are considered. The study is carried out in the hypotheses of an ideal incompressible fluid and a two-dimensional irrotational flow with gravity and surface tension effects being neglected.

When an initially floating body starts to move down into the water, an accurate description of the resulting hydrodynamic forces requires high order uniform initial asymptotics of the flow and the hydrodynamic pressure distribution [1]. If a body is flared, the first order solution is already singular about the intersection points, which makes the details of the flow about the intersection even more important.

At the previous workshop the starting flow generated by a floating wedge impact has been analyzed within a small time expansion procedure. Owing to the singularity occurring at the intersection point, the first order inner solution has been derived and matched with the corresponding outer one [2]. The inner solution was found to be strongly dependent on the value of the deadrise angle γ of the wedge, eventually leading, for $\gamma \leq \pi/4$, to eigensolutions orders of which are comparable with other terms retained in the expansions.

In order to evaluate the behavior of the hydrodynamic force at the initial stage of the impact, a small time expansion procedure is here developed up to the second order. The study is performed within the framework of the Lagrangian variables. The position of a fluid particle, the velocity potential and the local pressure are written as power series in time and substituted into the governing equations thus recovering boundary-value problems for the coefficients of the series [3]. Due to the Lagrangian formulation employed, the boundary value problems are defined onto a fixed domain, which is the same at any order of the expansion.

Since the first order outer solution is singular in this case, we follow the method of matched asymptotic expansions [4]. According to this method, the second order outer solution is recovered by enforcing its matching with the outer limit of the first order inner solution. It is found that the eigensolutions characterizing the inner solution for $\gamma \leq \pi/4$ require additional, non integer, power of time to be introduced into the series expansion of the outer solution. This implies that, for bodies with pronounced flares, details of the flow about the intersection points may significantly influence the pressure amplitude not only close to these points but also throughout the whole flow domain.

2. FORMULATION OF THE PROBLEM

The flow about a wedge initially floating on a still liquid surface is studied during the early stage of an impulsive vertical water entry. It is convenient to use non-dimensional variables taking some quantities L_* and V_* as the characteristic length scale and the characteristic scale of the flow velocity, respectively. In this case the pressure scale is $\rho_\ell V_*^2$, where ρ_ℓ is the liquid density. All variables and quantities used below are non-dimensionalized with respect to the chosen scales. Particular values of the scales L_* and V_* do not effect the solution procedure and are not specified here.

The entry velocity V is assumed to be constant after the initial impact. In the following γ denotes the deadrise angle, x the horizontal axis with $x = 0$ at the axis of symmetry of the body and y is the vertical axis oriented upward with $y = 0$ at the still free surface. Initially ($t = 0$) the wedge apex is at $y = -h_0$. The position of the body contour for $t > 0$ is described by the equation

$$y = |x| \tan\gamma - h_0 - Vt. \qquad (1)$$

The liquid is assumed ideal and incompressible and its flow irrotational and symmetric with respect to the axis $x = 0$. Gravity and surface tension effects are neglected. The flow is described by the velocity potential $\phi(x, y, t)$.

The study is carried out within the framework of Lagrangian variables ξ, η which identify a fluid particle position through the relations

$$x = x(\xi, \eta, t) \qquad y = y(\xi, \eta, t) \qquad (2)$$

with ξ and η being the horizontal and vertical coordinates of the particle at $t = 0$.

Following the approach by Korobkin & Wu [3], all unknown quantities are presented as power series of time and substituted into the governing equations. This procedure provides the boundary-value problems for the coefficients of the expansions. The mapping function from the $\boldsymbol{\xi} = (\xi, \eta)$ plane to the $\boldsymbol{x} = (x, y)$ one is written as

$$\boldsymbol{x}(\boldsymbol{\xi},t)=\boldsymbol{\xi}+H(t)\sum_{n=1}^{\infty}t^n\boldsymbol{X}^{(n)}(\boldsymbol{\xi}) \qquad , \tag{3}$$

where $H(t)$ is the Heaviside function and $\boldsymbol{X}^{(n)}$ are the unknown coefficients. It is convenient to introduce the new unknown function $\chi(\boldsymbol{\xi},t)=-\phi[\boldsymbol{x}(\boldsymbol{\xi},t),t]$. The small time expansions of the velocity potential and the hydrodynamic pressure are:

$$\chi(\boldsymbol{\xi},t) = H(t)\sum_{n=1}^{\infty}t^{n-1}\chi^{(n)}(\boldsymbol{\xi}) \qquad , \tag{4}$$

$$p(\boldsymbol{\xi},t) = p^{(1)}(\boldsymbol{\xi})\delta(t)+H(t)\sum_{n=2}^{\infty}t^{n-2}p^{(n)}(\boldsymbol{\xi}) \tag{5}$$

Substituting (3) - (5) into the equations of motion and the boundary conditions and collecting terms of the same order as $t\to 0$, we arrive at the boundary-value problems with respect to the unknown coefficients $\boldsymbol{X}^{(n)}(\boldsymbol{\xi})$, $\chi^{(n)}(\boldsymbol{\xi})$ and $p^{(n)}(\boldsymbol{\xi})$. In particular, equation (1) provides

$$\eta = \xi\tan\gamma-h_0, \tag{6}$$
$$Y^{(1)} = X^{(1)}\tan\gamma-V, \tag{7}$$
$$Y^{(2)} = X^{(2)}\tan\gamma \qquad . \tag{8}$$

From the definition of velocity $\boldsymbol{u}=(\phi_x,\phi_y)$, it follows

$$\boldsymbol{X}^{(1)}+\nabla\chi^{(1)}=0, \tag{9}$$
$$\boldsymbol{X}^{(2)}+(\nabla\boldsymbol{X}^{(1)})^T\cdot\boldsymbol{X}^{(1)}+\nabla\chi^{(2)}=0, \tag{10}$$

where $\nabla=(\partial_\xi,\partial_\eta)$. The continuity equation yields

$$\nabla\cdot\boldsymbol{X}^{(1)}=0, \tag{11}$$
$$\nabla\cdot\boldsymbol{X}^{(2)}+(\nabla X^{(1)})\cdot(\nabla^{\perp}Y^{(1)})=0, \tag{12}$$

where $\nabla^{\perp}=(\partial_\eta,-\partial_\xi)$. The Bernoulli's equation gives

$$p^{(1)}(\boldsymbol{\xi}) = \chi^{(1)}(\boldsymbol{\xi}), \tag{13}$$
$$p^{(2)}(\boldsymbol{\xi}) = \chi^{(2)}(\boldsymbol{\xi})+\frac{1}{2}\boldsymbol{X}^{(1)}\cdot\boldsymbol{X}^{(1)}. \tag{14}$$

Finally, the dynamic condition on the free surface

$$p^{(n)}(\boldsymbol{\xi})=0 \qquad (\eta=0,|\xi|>\xi_c), \tag{15}$$

where $\xi_c=h_0\cot\gamma$, and the far field condition

$$|\boldsymbol{X}^{(n)}|=O(|\boldsymbol{\xi}|^{-2}) \qquad (|\boldsymbol{\xi}|\to\infty) \tag{16}$$

have to be also satisfied.

3. DETERMINATION OF THE EXPANSION COEFFICIENTS

Starting from the equations derived above, the boundary value problems can be formulated for the first pressure coefficient $p^{(1)}(\xi,\eta)$

$$\begin{aligned}
\Delta p^{(1)} &= 0 \\
p^{(1)} &= 0 \qquad (\eta=0,|\xi|>\xi_c) \\
\partial_n p^{(1)} &= -V\cos\gamma \quad (\eta=|\xi|\tan\gamma-h_0,|\xi|<\xi_c) \\
p^{(1)} &\to 0 \qquad (|\boldsymbol{\xi}|\to\infty)
\end{aligned}$$

and for the new unknown function $q^{(2)}=p^{(2)}+\left(X^{(1)}\right)^2$

$$\begin{aligned}
\Delta q^{(2)} &= 0 \\
q^{(2)} &= 0 \qquad (\eta=0,|\xi|>\xi_c) \\
\partial_n q^{(2)} &= \partial_n\left(X^{(1)}\right)^2 \quad (\eta=|\xi|\tan\gamma-h_0,|\xi|<\xi_c) \\
q^{(2)} &\to 0 \qquad (|\boldsymbol{\xi}|\to\infty)
\end{aligned}$$

The solutions of these boundary-value problems are obtained with the help of the conformal mapping technique. To this aim the conformal mapping

$$z=-ih_0+\frac{l}{w}e^{i\gamma}\int_0^{\zeta}\left(\frac{\zeta_0^2}{1-\zeta_0^2}\right)^{\frac{\gamma}{\pi}}d\zeta_0 \tag{17}$$

is employed, which maps the lower half-plane $\zeta=\lambda+i\mu$, $\mu<0$, onto the fluid domain $z=\xi+i\eta$. Here $l=h_0/\sin\gamma$ and

$$w=\int_0^1\left(\frac{\zeta_0^2}{1-\zeta_0^2}\right)^{\frac{\gamma}{\pi}}d\zeta_0=\frac{1}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}+\frac{\gamma}{\pi}\right)\Gamma\left(1-\frac{\gamma}{\pi}\right).$$

Transformation (17) maps the body contour on the segment $\zeta=(\lambda,0)$ $|\lambda|<1$ and, correspondingly, the free surface on the two lines $\lambda<-1$, $\mu=0$ and $\lambda>1$, $\mu=0$.

By taking into account the far field behavior of the mapping function, it can be shown that the first order pressure coefficient is

$$p^{(1)}(z)=\Re\left\{-iV\left[z-\frac{l}{w}\sqrt{\zeta^2(z)-1}\right]\right\} \tag{18}$$

which, due to equation (13), coincides with the first order coefficient of the velocity potential. By using equation (9) the first order coefficients of the particle displacements can be recovered as follows:

$$X^{(1)}(z)=\Re\left\{iV\left[1-\left(\frac{\zeta^2}{\zeta^2(z)-1}\right)^{\frac{1}{2}-\frac{\gamma}{\pi}}\right]\right\} \tag{19}$$

$$Y^{(1)}(z)=-\Re\left\{V\left[1-\left(\frac{\zeta^2}{\zeta^2(z)-1}\right)^{\frac{1}{2}-\frac{\gamma}{\pi}}\right]\right\} \tag{20}$$

To determine the second order coefficient $q^{(2)}$, the Cauchy integral theorem is used. For $\gamma\neq\pi/4$ we obtain

$$Q^{(2)}(\zeta)=i\frac{C_O}{\sqrt{\zeta^2-1}}+ \tag{21}$$

$$+\frac{V^2\sin 2\gamma}{2\pi i\sqrt{\zeta^2-1}}\int_{-1}^{1}\frac{(s^2)^{1-\frac{2\gamma}{\pi}}(1-s^2)^{-\frac{1}{2}+\frac{2\gamma}{\pi}}}{s-\zeta}\mathrm{sign}(s)\,ds,$$

where $q^{(2)}(z)=\Re\{Q^{(2)}(\zeta[z])\}$. For $\gamma=\pi/4$ equation (21) takes a simpler form

$$Q^{(2)}(\zeta)=\frac{V^2\zeta}{2\pi i\sqrt{\zeta^2-1}}\ln\left(\frac{\zeta-1}{\zeta+1}\right)+i\frac{C_O}{\sqrt{\zeta^2-1}}. \tag{22}$$

Once $q^{(2)}$ has been found, the pressure coefficient $p^{(2)}$ follows from the definition of $q^{(2)}$ and then velocity potential $\chi^{(2)}$ is evaluated from equation (14).

Equations (21) and (22) show that the second order solution cannot be completely determined since the constant C_O is unknown. By following the method of the matched asymptotic expansion [4], this constant has to be calculated using the condition of matching between the inner limit of the second order outer solution and the outer limit of the first order inner solution. To this aim the expressions of the inner (at first order) and outer (at second order) velocity potential are written in the same set of spatial variables and then the matching is enforced. Since equation (21) is not integrable in a closed form, the matching procedure is considered in section 5 only in the case $\gamma = \pi/4$.

4. OUTER LIMIT OF THE FIRST ORDER INNER VELOCITY POTENTIAL

In [2] the first order inner solution and its asymptotic behavior have been recovered in terms of stretched variables φ, ρ, which are related to the corresponding physical ones by the relations

$$\phi(r,\theta,t) = Aa^{\sigma_0}(t)\varphi(\rho,\theta) \qquad r = a(t)\rho \tag{23}$$

where $\sigma_0 = \pi/2\beta$, $\beta = \pi - \gamma$,

$$A = V\left(\frac{h_0}{w\sin\gamma}\right)^{1-\sigma_0}\sigma_0^{-\sigma_0} \qquad ,$$

r, ρ denote the distance from the right-hand side intersection point, θ is the angular coordinate with $\theta = 0$ on the body contour and $\theta = \beta$ on the undisturbed free surface and $a(t)$ is the stretching function which is determined by enforcing the matching of the inner solution with the inner limit of the first order outer solution:

$$a(t) = [(2-\sigma_0)At]^{1/(2-\sigma_0)}.$$

To make the matching with the second order outer solution possible, the asymptotic behavior of the first order inner solution has to be recast in terms of the Lagrangian variables introduced above.

In the limit as $\rho \to \infty$, the inner velocity potential has been found to behave as

$$\begin{aligned}\varphi(\rho,\theta) &\simeq -\rho^{\frac{2}{3}}\cos\left(\frac{2}{3}\theta\right) - \frac{2}{9\pi}\theta\sin\left(\frac{2}{3}\theta\right)\rho^{-\frac{2}{3}} \\ &+ \left(C_I - \frac{2}{9\pi}\ln\rho\right)\cos\left(\frac{2}{3}\theta\right)\rho^{-\frac{2}{3}}\end{aligned} \tag{24}$$

for $\gamma = \pi/4$ and as

$$\begin{aligned}\varphi(\rho,\theta) &\simeq -\rho^{\sigma_0}\cos(\sigma_0\theta) + C_I\rho^{-\sigma_0}\cos(\sigma_0\theta) \\ &+ \frac{\sigma_0^2}{2-\sigma_0}\frac{\cos[2(1-\sigma_0)\theta]}{2\cos(2\gamma)}\rho^{2(\sigma_0-1)}\end{aligned} \tag{25}$$

for $\gamma \neq \pi/4$. As long as $\gamma > \pi/4$, the eigensolution term in (25) is of higher order with respect to the other contributions and can be neglected.

In terms of outer variables equations (24) and (25) take the forms

$$\begin{aligned}\phi_I(r,\theta,t) &= -Ar^{\frac{2}{3}}\cos\left(\frac{2}{3}\theta\right) + \frac{2}{9\pi}r^{-\frac{2}{3}}\cos\left(\frac{2}{3}\theta\right) \\ & \quad A^2 t\ln\left(\frac{4}{3}At\right) + A^2\left[\frac{4}{3}C_I\cos\left(\frac{2}{3}\theta\right)r^{-\frac{2}{3}}\right. \\ &- \frac{8}{27\pi}\theta\sin\left(\frac{2}{3}\theta\right)r^{-\frac{2}{3}} \\ &- \left.\frac{8}{27\pi}\cos\left(\frac{2}{3}\theta\right)r^{-\frac{2}{3}}\ln r\right]t\end{aligned} \tag{26}$$

and

$$\begin{aligned}\phi_I(r,\theta,t) &= A\left\{-r^{\sigma_0}\cos(\sigma_0\theta)\right. \\ &+ C_I[(2-\sigma_0)A]^{\frac{2\sigma_0}{2-\sigma_0}}r^{-\sigma_0}\cos(\sigma_0\theta)t^{\frac{2\sigma_0}{2-\sigma_0}} \\ &+ \left.\frac{\cos[2(1-\sigma_0)\theta]}{2\cos(2\gamma)}\sigma_0^2 r^{2(\sigma_0-1)}At\right\}\end{aligned} \tag{27}$$

respectively. Equations (26) and (27) immediately show that the small time expansion with integer power of time employed in the equations (3)-(5) does not allow the matching with the inner solution and, instead, logarithmic time dependent terms ($\gamma = \pi/4$) or non integer power of time ($\gamma < \pi/4$) have to be introduced. Through this mechanism, the details of the flow in the jet region affect in a substantial manner the dynamics of the solution in a quite wider region, at least for deadrise angle equal or smaller than $\pi/4$.

5. DEADRISE ANGLE OF 45 DEGREES

It is convenient to introduce the complex variable

$$\tau = r\exp(i\theta)$$

with the help of which expansion (26) can be presented as

$$\phi_I(r,\theta,t) = \Re\Big[-A\tau^{2/3} + t\ln t\Big(\frac{2A^2}{9\pi}\Big)\tau^{-2/3} + \tag{28}$$

$$t\Big(\frac{2A^2}{27\pi}\Big)\tau^{-2/3}\Big(3\ln(\frac{4A}{3}) + 18C_I - 4\ln\tau\Big)\Big].$$

In the leading order as $t \to 0$ we obtain that $z - x_c \approx \tau\exp(-i\beta)$ and (17) gives

$$z - x_c = \frac{h_0}{w\sin\gamma}\frac{\pi}{\beta 2^{\gamma/\pi}}[\rho\exp(i(\pi+\omega))]^{\beta/\pi} + ...$$

$$\zeta = 1 - \rho\exp(i\omega), \quad 0 \le \omega \le \pi$$

close to the intersection point, where $|\zeta - 1| \ll 1$. Combining the latter equations, we find

$$\frac{2A^2}{9\pi}\tau^{-2/3} \approx \frac{V^2}{6\pi i\sqrt{\zeta^2-1}},$$

which makes it possible to rewrite (28) as

$$\phi_I(r,\theta,t) = \Re\Big[-A\tau^{2/3}(\zeta,t) + t\ln t\frac{-V^2 i}{2\pi\sqrt{\zeta^2-1}} \tag{29}$$

$$+t\Big\{-\frac{V^2\zeta}{2\pi i\sqrt{\zeta^2-1}}\ln\Big(\frac{\zeta-1}{\zeta+1}\Big)$$

$$-\frac{V^2 i}{2\pi\sqrt{\zeta^2-1}}[\ln(3Vw/4l)+6C_I]+\frac{V^2}{2\sqrt{\zeta^2-1}}\Big\}.$$

Comparing (22) and (29), one may observe that the first term in (22) matches the asymptotic expansion (29). It should be noted that the first term in (29) requires further expansion as $t\to 0$ and $\zeta\to 1$ but it does not give a contribution of the order $O(t\ln t)$.

The outer limit of the inner solution (29) indicates that the small time expansion of the outer solution in the case $\gamma=\pi/4$ is different from (3) - (5) and has the form

$$\boldsymbol{x}(\boldsymbol{\xi},t)=\boldsymbol{\xi}+t\boldsymbol{X}^{(1)}+t^2\ln t\boldsymbol{X}^{(2)}+t^2\boldsymbol{X}^{(3)}+o(t^2),\quad(30)$$

$$\chi(\boldsymbol{\xi},t)=\chi^{(1)}+t\ln t\chi^{(2)}+t\chi^{(3)}(\boldsymbol{\xi})+o(t),\qquad(31)$$

$$p(\boldsymbol{\xi},t)=p^{(1)}\delta(t)+\ln tp^{(2)}+p^{(3)}+o(1).\qquad(32)$$

The coefficients $\boldsymbol{X}^{(1)}(\boldsymbol{\xi})$, $\chi^{(1)}(\boldsymbol{\xi})$ and $p^{(1)}(\boldsymbol{\xi})$ were determined in section 3. The functions $\boldsymbol{X}^{(2)}(\boldsymbol{\xi})$, $\chi^{(2)}(\boldsymbol{\xi})$ and $p^{(2)}(\boldsymbol{\xi})$ represent the eigensolution of the outer problem. They are determined using the procedure similar to that described in section 2. We find

$$p^{(2)}(\boldsymbol{\xi})=\chi^{(2)}(\boldsymbol{\xi}),\quad \boldsymbol{X}^{(2)}(\boldsymbol{\xi})=-\frac{1}{2}\nabla p^{(2)},$$

where $p^{(2)}(\boldsymbol{\xi})$ is the harmonic function which satisfies homogeneous boundary conditions. The amplitude of the eigensolution is obtained by its matching with the inner solution (29). By algebra

$$\chi^{(2)}(\boldsymbol{\xi},t)=\Re\Big[\frac{V^2 i}{2\pi\sqrt{\zeta^2(z)-1}}\Big]\qquad(33)$$

and, in particular,

$$p^{(2)}(\lambda,0)=-\frac{V^2}{2\pi\sqrt{1-\lambda^2}}\quad(|\lambda|<1).\qquad(34)$$

Thus we arrive at the following small time expansion of the hydrodynamic pressure along the wetted part of the floating wedge

$$p(\xi,\eta,t)=p^{(1)}(\xi,\eta)\delta(t)+\ln\Big(\frac{1}{t}\Big)\frac{V^2}{2\pi\sqrt{1-\lambda^2}}+O(1).\qquad(35)$$

The first term in (35) is given by the pressure-impulse theory (see section 2) but the second one reflects the influence of the inner solution on the outer pressure distribution. Equation (35) can be used to evaluate the initial asymptotics of the hydrodynamic force acting on the floating wedge.

6. HYDRODYNAMIC FORCE

Analysis of the inner solution provides that its contribution to the hydrodynamic force is of the order of $O(t^s)$ as $t\to 0$, where $s=2\gamma/(3\pi-4\gamma)$. Therefore, up to the order of $O(1)$ the asymptotics of the hydrodynamic force $F(t)$ is determined by the outer solution. In the dimensional variables

$$F(t)=2\rho_\ell V_*^2L_*[\int_0^{x_c}p(x,y(x),t)dx+o(1)].\qquad(36)$$

Here $x=\xi+O(t)$ and asymptotics of the pressure is given by (35). Taking $V_*=V$ and $L_*=h_0$, we obtain in the case $\gamma=\pi/4$

$$F(t)=2\rho_\ell V^2h_0[F_0\delta(t)+F_1\ln\Big(\frac{1}{t}\Big)+O(1)],\qquad(37)$$

Calculations yield

$$F_0=\frac{1}{2}\pi^2\Gamma^{-4}(3/4)-1,$$

$$F_1=\frac{1}{2\pi w}\int_0^1\frac{\lambda^{1/2}d\lambda}{(1-\lambda^2)^{3/4}}=\frac{1}{2}\sqrt{\pi/2}\Gamma^{-2}(3/4).$$

The initial asymptotics of the hydrodynamic force on a floating wedge with its deadrise angle of 45 degrees is given as

$$F(t)=2\rho_\ell V^2h_0[0.786\delta(t)+0.417\ln\Big(\frac{1}{t}\Big)+O(1)].\quad(38)$$

It is important to notice that this asymptotics is determined by the outer solution which is strongly affected by details of the flow close to the intersection points for bodies with pronounced flares.

The following term in (38) can be also evaluated using the present results and the matching procedure. Similar formulae can also be derived in the case $\gamma\neq\pi/4$. In order to do this, asymptotics of the integral term in (21) as $|\zeta-1|\ll 1$ has to be used.

7. ACKNOWLEDGMENTS

This work has been carried out during the visit of A.I. at the L.I.H. in September 2001 in the framework of the Short-term Mobility Program financially supported by the CNR (*Consiglio Nazionale delle Ricerche*).
A.A.K. acknowledges the support from RFBR (projects No. 00-01-00839 and No. 00-15-96162)

8. REFERENCES

1 T. Vinje, 'On small-time expansion of nonlinear free surface problems', J. Engng. Math., **28**, 173-190, 1994.

2 A. Iafrati, A.A. Korobkin 'Starting flow generated by a floating wedge impact. Part 1: Asymptotic analysis', submitted to J. Fluid Mech, 2001.

3 A.A. Korobkin, G.X. Wu, *Impact on a floating circular cylinder*, Proc. Royal Soc.: Mathematical, Physical and Engineering sciences, **456**, 2489-2514, 2000.

4 A.H. Nayfeh, 'Perturbation Methods' A Wiley-Interscience Publication, NY, 1976.