SECOND-ORDER DIFFRACTION OF A UNIDIRECTIONAL WAVE GROUP

E. V. Buldakov, P. H. Taylor and R. Eatock Taylor

Department of Engineering Science, University of Oxford, Parks Road, Oxford OX1 3PJ, UK

SUMMARY

The problem of the diffraction of an unidirectional incident wave group by a bottom-seated cylinder is considered. The problem is formulated for two first approximations of a small perturbation theory. It is solved by using the time Fourier transform and variables separation. Solutions for various types of incoming wave spectrum (Gaussian and Pierson-Moskowitz) presented, and the solution technique is optimized for wave group type problems.

1 PROBLEM FORMULATION

We consider diffraction of an unidirectional wave in water of uniform depth $\tilde{h}$ by a bottom-seated cylinder of radius $\tilde{a}$. The intersection point of the cylinder axis and an undisturbed water surface is the origin of a rectangular Cartesian coordinate system $(\tilde{x}, \tilde{y}, \tilde{z})$ and cylindrical coordinates $(\tilde{r}, \theta, \tilde{z})$. A unidirectional wave-group with surface elevation $\tilde{\Xi} = \tilde{A}\,\Xi_I$ and characteristic amplitude $\tilde{A}$ is generated at $\tilde{x} = -\infty$ and moves in the positive x-direction. We introduce non-dimensional variables in the following way

$$(\tilde{x}, \tilde{y}, \tilde{z}) = \tilde{L}\ (x, y, z); \quad \tilde{\Xi} = \tilde{A}\,\Xi; \quad \tilde{t} = \sqrt{\tilde{L}/\tilde{g}}\ t; \quad \tilde{\Phi} = \tilde{A}\,\tilde{L}\,\tilde{g}^{1/2}\ \Phi,$$

where $\tilde{\omega}_0$ is the incoming wave characteristic frequency. A tilde is used to denote dimensional values. The problem formulation becomes

$$\begin{gathered}\Delta\Phi = 0; \qquad \frac{\partial \Phi}{\partial z} = 0\Big|_{z=-h}; \quad \frac{\partial \Phi}{\partial r} = 0\Big|_{r=1}; \\ \frac{\partial^2 \Phi}{\partial t^2} + \frac{\partial \Phi}{\partial z} = -\varepsilon\,\frac{\partial}{\partial t}(\nabla\Phi)^2 - \varepsilon^2\,\frac{1}{2}\,\nabla\Phi\,\nabla(\nabla\Phi)^2\Big|_{z=\varepsilon\,\Xi}; \quad \Xi(x,y,t) = -\frac{\partial \Phi}{\partial t} - \varepsilon\,\frac{1}{2}(\nabla\Phi)^2\Big|_{z=\varepsilon\,\Xi}\end{gathered} \tag{1}$$

and includes one non-dimensional parameter $\varepsilon = \tilde{A}/\tilde{L}$. Here the length scale $\tilde{L} = \tilde{a}$ and $\varepsilon = \tilde{A}/\tilde{a}$. We consider the limit $\varepsilon \to 0$ and represent the solution as an asymptotic expansion

$$\Phi = (\Phi^{(1)} + \Phi^{(1)^*}) + \varepsilon\,(\Phi^{(2)} + \Phi^{(2)^*}) + \cdots; \quad \Xi = (\Xi^{(1)} + \Xi^{(1)^*}) + \varepsilon\,(\Xi^{(2)} + \Xi^{(2)^*}) + \cdots. \tag{2}$$

The problem has a second length scale, a characteristic length of the incoming wave, and the measure of non-linearity for the incoming wave is the linear steepness $\varepsilon_I = \tilde{A}\,\tilde{\omega}_0^2/\tilde{g}$. Here we assume that ε_I and ε are of the same order, $\varepsilon_I = O(\varepsilon)$ as $\varepsilon \to 0$. Applying Fourier transform to separate time explicitly we obtain

$$\begin{gathered}\Delta\phi_D^{(1,2)} = 0; \qquad \frac{\partial \phi_D^{(1,2)}}{\partial z} = 0\Big|_{z=-h}; \quad \frac{\partial \phi_D^{(1,2)}}{\partial r} = -\frac{\partial \phi_I^{(1,2)}}{\partial r}\Big|_{r=1}; \\ -\,\omega^2\,\phi_D^{(1,2)} + \frac{\partial \phi_D^{(1,2)}}{\partial z} = rhs^{(1,2)}\Big|_{z=0}; \quad \xi_D^{(1,2)} = i\,\omega\,\phi_D^{(1,2)} + se^{(1,2)}\Big|_{z=0},\end{gathered} \tag{3}$$

where

$$rhs^{(1)} = 0; \quad se^{(1)} = 0; \quad rhs^{(2)} = rhs^+ + rhs^- - rhs_I; \quad se^{(2)} = se^+ + se^- - se_I$$

and

$$rhs^+ = \frac{i}{\sqrt{2\,\pi}}\int_{-\infty}^{+\infty}\left(2\,s\,\nabla\phi^{(1)}(s)\,\nabla\phi^{(1)}(\omega-s) + s^2\,(\omega-s)\,\frac{\partial\phi^{(1)}(s)}{\partial z}\,\phi^{(1)}(\omega-s) - (\omega-s)\,\frac{\partial^2\phi^{(1)}(s)}{\partial z^2}\,\phi^{(1)}(\omega-s)\right)ds\Big|_{z=0};$$

$$rhs^- = \frac{i}{\sqrt{2\,\pi}}\int_{-\infty}^{+\infty}\left(2\,s\,\nabla\phi^{(1)}(s)\,\nabla\phi^{(1)^*}(s-\omega) + s^2\,(\omega-s)\,\frac{\partial\phi^{(1)}(s)}{\partial z}\,\phi^{(1)^*}(s-\omega) - (\omega-s)\,\frac{\partial^2\phi^{(1)}(s)}{\partial z^2}\,\phi^{(1)^*}(s-\omega)\right)ds\Big|_{z=0};$$

$$se^+ = -\frac{1}{\sqrt{2\,\pi}}\int_{-\infty}^{+\infty}\left(\frac{1}{2}\,\nabla\phi^{(1)}(s)\,\nabla\phi^{(1)}(\omega-s) + s\,(\omega-s)\frac{\partial\phi^{(1)}(s)}{\partial z}\phi^{(1)}(\omega-s)\right)ds\Big|_{z=0};$$

$$se^- = -\frac{1}{\sqrt{2\,\pi}}\int_{-\infty}^{+\infty}\left(\frac{1}{2}\,\nabla\phi^{(1)}(s)\,\nabla\phi^{(1)^*}(s-\omega) + s\,(\omega-s)\frac{\partial\phi^{(1)}(s)}{\partial z}\phi^{(1)^*}(s-\omega)\right)ds\Big|_{z=0}.$$

Lower case letters here denote the corresponding Fourier transforms, and subscripts I and D designate the incoming and diffracted components respectively. The corresponding first and second-order problems are the same as for a monochromatic incoming wave (e. g. Chau & Eatock Taylor, 1992) save for the form of the non-homogeneous terms in the second-order problem. The amplitude of the solution for every specific value of ω is specified by the spectrum of the incoming wave.

A general linear unidirectional wave of the first-order solution approaching from $x = -\infty$ in the positive direction can be represented as a linear combination of monochromatic sinusoidal waves

$$\Xi_I^{(1)}(x,t) = \frac{1}{2}\int_{-\infty}^{+\infty} S(\Omega)\, e^{i\,(\,k(\Omega)\,x-\Omega\,t+\varphi(\Omega)\,)}\, d\Omega, \tag{4}$$

where the spectrum S is defined such that at the focus point of the wavegroup ($t = 0$, $x = 0$) the surface elevation is $\Xi_I^{(1)}(0,0)+\Xi_I^{(1)^*}(0,0) = 1$. The non-dimensional wave number k here satisfies the dispersion relation $\omega^2 = k(\omega)\,\tanh(k(\omega)\,h)$. The corresponding first-order potential is

$$\Phi_I^{(1)} = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{S(\Omega)}{i\,\Omega}\,\frac{\cosh(k(\Omega)\,(z+h))}{\cosh(k(\Omega)\,h)}\, e^{i\,(\,k(\Omega)\,x-\Omega\,t+\varphi(\Omega)\,)}\, d\Omega.$$

Introducing a complex amplitude function $A(\omega) = -i\,\sqrt{\pi/2}\,e^{i\,\varphi(\omega)}\,S(\omega)/\omega$ we can write the Fourier component of the incoming wave at a frequency ω as

$$\phi_I^{(1)}(\omega) = A(\omega)\,\frac{\cosh(k(\omega)\,(z+h))}{\cosh(k(\omega)\,h)}\,e^{i\,k(\omega)\,x}.$$

The corresponding second-order solution is

$$\phi_I^{(2)} = \phi_I^{+} + \phi_I^{-}; \quad \phi_I^{\pm} = \int_{-\infty}^{+\infty} \frac{\mathbb{A}^{\pm}(s)\,\cosh(\kappa^{\pm}(z+h))}{\kappa^{\pm}\,\sinh(\kappa^{\pm}\,h) - \omega^2\cosh(\kappa^{\pm}\,h)}\,e^{i\,\kappa^{\pm}x}\,ds, \tag{5}$$

where $\kappa^{\pm} = k(s) \pm k(\omega - s)$ and

$$\mathbb{A}^{+}(s) = \frac{i}{\sqrt{2\,\pi}}\,(\,-2\,s\,k(s)\,k(\omega-s) + (\omega-s)\,(2\,s^3\,(\omega-s) + s^4 - k^2(s))\,)\,A(s)\,A(\omega-s);$$

$$\mathbb{A}^{-}(s) = \frac{i}{\sqrt{2\,\pi}}\,(\,2\,s\,k(s)\,k(\omega-s) + (\omega-s)\,(2\,s^3\,(\omega-s) + s^4 - k^2(s))\,)\,A(s)\,A^{*}(s-\omega).$$

The solution for the incoming as well as for the diffracted potentials includes two terms. These are the plus-term ϕ^{+} corresponding to the sums of various frequencies and the minus-term ϕ^{-} which corresponds to their differences. For the case of single frequency input these are double frequency and constant second-order terms respectively.

2 SOLUTION FOR THE DIFFRACTED POTENTIALS

To solve the non-homogeneous second-order problem (3) let us first consider the following eigenvalue problem in the vertical direction

$$Z''_m(z) - \lambda_m^2\,Z_m(z) = 0; \quad Z'_m(-h) = 0; \quad Z'_m(0) - \omega^2\,Z_m(0) = 0. \tag{6}$$

The problem (6) has a countable set of eigenvalues and eigensolutions

$$\lambda_0 = k_0; \qquad \omega^2 = k_0\tanh(k_0h); \qquad Z_0(z) = \frac{\cosh(k_0(z+h))}{\cosh(k_0h)}$$

$$\lambda_m = i\,k_m; \quad \omega^2 = -k_m\tan(k_mh); \quad Z_m(z) = \frac{\cos(k_m(z+h))}{\cos(k_mh)}$$

The system of functions Z_m is orthogonal on the interval $[-h,0]$. We multiply the second-order problem (3) by $Z_m(z)$ and integrate the result from $-h$ to 0. For the second derivative with respect to z in the Laplacian we have

$$\int_{-h}^{0} Z_m(z)\frac{\partial^2\phi_D^{(2)}}{\partial z^2}\,dz = Z_m\frac{\partial\phi_D^{(2)}}{\partial z}\bigg|_{-h}^{0} - Z'_m\phi_D^{(2)}\bigg|_{-h}^{0} + \int_{-h}^{0} Z''_m\phi_D^{(2)}\,dz = rhs^{(2)}(x,y) \pm k_m^2\int_{-h}^{0} Z_m\phi_D^{(2)}\,dz,$$

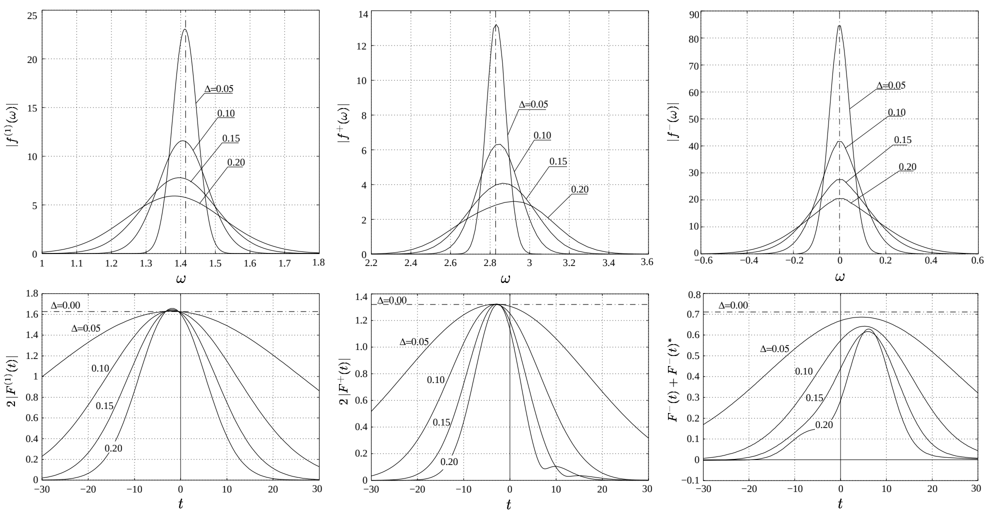

Figure 1: First-order and second-order plus and minus force amplitude spectrum $|f(\omega)|$ and force time history $F(t)$ for various bandwidths of Gaussian incoming wave spectrum, $\omega_0 = \sqrt{2}$. The dash-dotted line represents the monochromatic result from Kim & Yue (1989).

where the free-surface boundary condition was used. Now the functions

$$\psi_m(x,y) = \int_{-h}^{0} Z_m(z)\phi_D^{(2)}(x,y,z)\,dz$$

satisfy the following set of the boundary value problems in the (x,y) plane

$$\begin{aligned} &\Delta\psi_0 + k_0^2\psi_0 + rhs^{(2)}(x,y) = 0; \\ &\Delta\psi_m - k_m^2\psi_0 + rhs^{(2)}(x,y) = 0; \end{aligned} \qquad \left.\frac{\partial\psi_m}{\partial r}\right|_{r=1} = -\int_{-h}^{0} Z_m \left.\frac{\partial\phi_I^{(2)}}{\partial r}\right|_{r=1} dz \tag{7}$$

plus the appropriate radiation condition at infinity. The complete second order solution can be now written in the form of the Fourier series in the vertical direction

$$\phi_D^{(2)} = \sum_{m=0}^{\infty} b_m Z_m(z)\psi_m(x,y), \quad b_m = \left(\int_{-h}^{0} Z_m(z)^2\,dz\right)^{-1}.$$

We shall represent the solution of (7) in the form of the Fourier series in the circumferential direction

$$\psi_m(r,\theta) = \sum_{n=-\infty}^{\infty} R_{nm}(r)\,e^{in\theta},$$

where each term of the series satisfies the boundary value problem for a non-homogeneous Bessel equation

$$R''_{nm} + \frac{1}{r}R'_{nm} + (\pm k_m^2 - \frac{n^2}{r^2})R_{nm} = -rhs_n^{(2)}(r); \quad R'_{nm}(1) = -R'_{Imn}(1) \tag{8}$$

with proper boundary condition at infinity. The far field asymptotic behaviour of the individual term of the Fourier series of the plus component of the right hand side is

$$rhs_n^+ \to \frac{1}{r^{3/2}} A_n \exp(2\,i\,k(\frac{\omega}{2})\,r) \quad \text{as} \quad r \to \infty$$

while the corresponding minus component is exponentially small.

After substitution in (8) it can be easily found that the non-homogeneous part of the plus-solution (locked wave) behaves like

$$R_{L\,nm}^{+}(r) \to -\frac{A_n}{\pm k_m^2 - 4\,k(\omega/2)^2}\,\frac{1}{r^{3/2}}\,e^{2\,i\,k(\omega/2)\,r}$$

and the minus-term decays exponentially. The homogeneous part (free wave) satisfying the Helmholtz radiation condition can be represented as

$$R_{F\,nm}^{\pm}(r) \to \frac{C_{nm}}{\sqrt{r}}\,e^{ik_m r} \quad \text{as} \quad r \to \infty$$

for $m = 0$, and it decays exponentially for $m > 0$. To complete the formulation of the problem for the second order diffracted Fourier components, we use these conditions as an asymptotic boundary condition for problem (8).

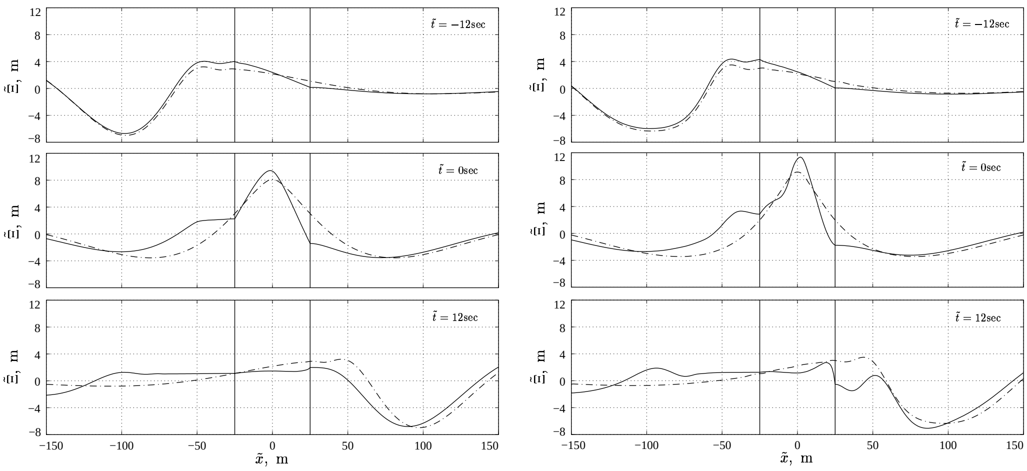

Figure 2: Snapshots of the first order (left) and complete first plus second order (right) dimensional surface elevation along the x-axis and around the circumference of the cylinder. Pierson-Moskowitz spectrum, $\tilde{A} = 8.15\,\mathrm{m}$, $\tilde{h} = 50\,\mathrm{m}$, $\tilde{a} = 25\,\mathrm{m}$, $\tilde{T}_0 = 12\,\mathrm{sec}$. The dash-dotted line represents the solution for the incoming wave. $\tilde{T}_0$ is the period corresponding to the frequency $\tilde{\omega}_0$.

3 RESULTS

As a first trial of the method we applied it to the case of a Gaussian spectrum for the incoming wave

$$S(\omega) = e^{-(\omega-\omega_0)^2/\Delta^2}/(\sqrt{\pi}\,\Delta)\,. \tag{9}$$

The limit of the spectrum (9) as $\Delta \to 0$ is a monochromatic wave. Calculations with $\Delta = 0.2, 0.15, 0.1$ and 0.05 for the basic frequency $\omega_0 = \sqrt{2}$ were performed.

Spectra of the force amplitude and the corresponding force time histories have been obtained. The force $F(t)$ has been decomposed into first-order and second-order plus and minus terms similarly to potential in (2). Figure 1 shows the envelopes of the first-order term $F^{(1)}$ and the plus term F^{+}, oscillating at frequencies ω_0 and $2\,\omega_0$, and the time history of the slowly varying minus term F^{-}. As the bandwidth of the original incoming wave spectrum Δ is decreased, all response spectra approach certain delta-type spectra with frequencies ω_0 for the first-order solution, $2\,\omega_0$ for the plus-solution and 0 for the minus-solution. The solutions with wider input spectra decay faster in time, which means that the corresponding wave envelopes are smaller in length and interact with the cylinder during the shorter time period. The solutions with smaller bandwidth approach the monochromatic solution of Kim & Yue (1989).

Next, we consider the more practical case of an incoming wave described by the Pierson-Moskowitz spectrum

$$S(\omega) = \frac{5}{\omega_0}\left(\frac{\omega}{\omega_0}\right)^{-5}\exp\left(-\frac{5}{4}\left(\frac{\omega}{\omega_0}\right)^{-4}\right).$$

Let the physical dimensional parameters of the problem be: amplitude $\tilde{A} = 8.15\,\mathrm{m}$, depth $\tilde{h} = 50\,\mathrm{m}$, cylinder radius $\tilde{a} = 25\,\mathrm{m}$ and the characteristic period of the wave $\tilde{T}_0 = 12\,\mathrm{sec}$. Then the corresponding non-dimensional parameters are $h = 2$, $\omega_0 = 0.836$ and $\varepsilon = .326$. The comparison of the first and the complete second-order solutions for this parameter set is represented on figure 2 where the snapshots of water surface elevation at three different times for both cases are shown. Since the parameter ε is relatively large for the case considered, the influence of the non-linearity in the diffracted field is considerable as can be clearly seen on the figure. It leads to high local surface gradients around the circumference of the cylinder.

In conclusion, the method proposed here works well for the interaction of compact wavegroups with a single cylinder.

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References

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Discussion Sheet

Abstract Title :	Second-order diffraction of unidirectional wave group		
(Or) Proceedings Paper No. :	05	***Page :***	017
First Author :	Buldakov, E.V.		
Discusser :	Paul Sclavounous		

Questions / Comments :

Michael Longuett-Higgins in the sixties showed that the consistent spectral density to second order requires the solution of up to the third-order problem. I extended this study in a JFM article where I solved the reflection of a random wave train off a vertical wall. Have you considered studying the third-order problem around a vertical circular cyclinder to extract consistent wave elevation and force statistics?

Author's Reply :
(If Available)

Author did not respond.