Verification of Fourier-Kochin representation of waves

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The purpose of this study is to present a verification of the Fourier-Kochin representation of waves given in [1,2]. This representation expresses the waves generated by a given flow at a boundary surface in terms of single Fourier integrals and spectrum functions that are defined by distributions of elementary waves over the boundary surface. The Fourier-Kochin representation of waves is given in [1,2] for three classes of free-surface flows: (i) diffraction-radiation of time-harmonic waves without forward speed, (ii) steady ship waves, and (iii) time-harmonic wave (diffraction-radiation with forward speed).

The Fourier-Kochin representation of waves is considered here for steady flows associated with the linearized free-surface boundary condition \( w + F^2 \partial u / \partial x = 0 \) where \( F = U / \sqrt{gL} \) is the Froude number, and \((u, v, w) = \tilde{u} = \tilde{U} / U = \nabla \phi \) is the disturbance-flow velocity; here, \( \phi = \Phi / (U L) \) is the velocity potential associated with the velocity \( \tilde{u} \). The Fourier-Kochin representation of waves defines the potential \( \phi^W \) and the velocity \( \tilde{u}^W \) associated with the waves that are generated by a given velocity distribution \( \tilde{u} \) at a boundary surface \( \Sigma \), which may intersect the mean free-surface plane \( z = 0 \) along the boundary curve \( \Gamma \). The boundary surface \( \Sigma \cup \Gamma \) is divided into patches, i.e. \( \Sigma \cup \Gamma = \sum_{p=1}^{N} \Sigma_p \cup \Gamma_p \), associated with reference points \((x_p, y_p, z_p)\), with \( \tilde{x} = \tilde{X} / L \), located near the centroids of the patches.

The wave potential \( \phi^W \) and velocity \( \tilde{u}^W \) at a field point \((\xi, \eta, \zeta)\) of the flow domain outside a boundary surface \( \Sigma \cup \Gamma \) are given by the single Fourier integrals

\[
4\pi \begin{pmatrix} \phi^W \\ u^W \\ w^W \end{pmatrix} = \mathcal{R} \Re \int_{-\infty}^{\infty} \frac{d\beta}{k^d - \nu} \begin{pmatrix} i \alpha^d \\ \beta \\ i k^d \end{pmatrix} \sum_{p=1}^{N} \left[ 1 + \text{erf} \left( \frac{x_p - \xi}{\sigma F^2 C} \right) \right] S_p^W e^{i(z_p + \zeta)k^d + i[(x_p - \xi)\alpha^d + (y_p - \eta)\beta]}
\]

where \( \mathcal{R} \) stands for the real part. The functions \( \alpha^d(\beta) \) and \( k^d(\beta) \) are defined as

\[
\alpha^d = \sqrt{k^d / F} \quad k^d = \nu + \sqrt{\nu^2 + \beta^2} \quad \text{with} \quad \nu = 1/(2F^2)
\]

Here, \( k^d(\beta) \) stands for the value of the wavenumber \( k \) at the dispersion curves \( \alpha = \pm \alpha^d(\beta) \), with \( -\infty \leq \beta \leq \infty \), associated with the dispersion relation \( F^2 \alpha^2 - k = 0 \). The function \( C \) in the error function \( \text{erf} \) is related to the curvature of the dispersion curves and is given by

\[
C = 1 + 3/(F^2 k^d) - 2 / (4 F^2 k^d - 3)^{3/2}
\]

We have \( C = 2 \) for \( \beta = 0 \), where \( \alpha^d = k^d = 1/F \), \( C = 1 \) as \( \beta \rightarrow \pm \infty \), and \( C = 1 \) at the inflection points defined by \( F^2 k^d = 3/2 \) and \( F^2 \beta = \pm \sqrt{3}/2 \). The positive real constant \( \sigma \) may be chosen as in [2].

The contribution \( S_p^W \) of patch \( p \) to the wave-spectrum function \( S^W(\beta) \) is given by

\[
S_p^W = S_p^\Sigma + F^2 S_p^\Gamma
\]

with

\[
S_p^\Sigma = \int_{\Sigma_p} dA \left[ (\tilde{u} \cdot \tilde{n}) + i \frac{\alpha^d}{k^d} (\tilde{u} \times \tilde{n}) \cdot \tilde{v} - \frac{\beta}{k^d} (\tilde{u} \times \tilde{n}) \cdot \tilde{v} \right] e^{k^d(z + z_p) + i[(\alpha^d(x - x_p) + \beta(y - y_p)]}
\]

\[
S_p^\Gamma = \int_{\Gamma_p} d\Gamma \left[ (t^2 \tilde{t} + \frac{\alpha^d}{k^d} \tilde{v} \cdot \tilde{t} - (t^2 + \tilde{v}^2) \tilde{u} \cdot \tilde{v} \right] e^{i[(\alpha^d(x - x_p) + \beta(y - y_p)]}
\]

Here, the unit vector \( \tilde{n} = (n_x, n_y, n_z) \) is normal to the boundary surface \( \Sigma \) and points into the flow region outside \( \Sigma \), and the unit vectors \( \tilde{t} = (t_x, t_y, 0) \) and \( \tilde{v} = (-t_y, t_x, 0) \) are tangent and normal to the boundary curve \( \Gamma \) in the mean free-surface plane \( z = 0 \). The normal vector \( \tilde{v} \) points into the flow region outside \( \Gamma \), like the normal vector \( \tilde{n} \), and the tangent vector \( \tilde{t} \) is oriented clockwise (looking down). The spectrum functions \( S^\Sigma(\beta) \) and \( S^\Gamma(\beta) \) are defined by distributions of elementary waves over the boundary surface \( \Sigma \) and the boundary curve \( \Gamma \), respectively, with amplitudes given by the normal components \( \tilde{u} \cdot \tilde{n}, \tilde{u} \cdot \tilde{v} \) and the tangential components \( \tilde{u} \times \tilde{n}, \tilde{u} \cdot \tilde{t} \) of the velocity \( \tilde{u} \) at \( \Sigma \) and \( \Gamma \).
Thus, the Fourier-Kochin wave representation defines the wave potential $\phi^W(\xi)$ and velocity $u^W(\xi)$ at a field point $\xi$ of the flow region outside a boundary surface $\Sigma \cup \Gamma$ in terms of the velocity distribution $\bar{u}(\bar{x})$ at the boundary surface $\Sigma$ and the boundary curve $\Gamma$. This representation of the waves generated by a flow at a boundary surface only involves the boundary velocity $\bar{u}(\bar{x})$; i.e. the Fourier-Kochin wave representation does not involve the potential $\phi(\bar{x})$ at the boundary surface $\Sigma \cup \Gamma$, unlike the classical boundary-integral representation that defines the potential in a potential-flow region in terms of boundary-values of the potential $\phi$ and its normal derivative $\partial \phi / \partial n = \bar{u} \cdot \bar{n}$. The Fourier-Kochin wave representation is based on several recent new fundamental results obtained within the framework of the Fourier-Kochin theory [3,2]: (i) the boundary-integral representation, called velocity representation, given in [1,2], (ii) the representation of the generic super Green function defined in [4,5,2], and (iii) the transformations of spectrum functions given in [3,1,2]. The flow generated by a given flow at a boundary surface can be expressed as

$$\phi = \phi^W + \phi^L, \quad \bar{u} = u^W + \bar{u}^L,$$

where $\phi^W$, $u^W$ is the wave component defined by the Fourier-Kochin wave representation, and $\phi^L$, $u^L$ is a local-flow component. The Rankine and Fourier-Kochin nearfield flow representation given in [6] expresses the local component $\phi^L$, $\bar{u}^L$ in terms of distributions of elementary Rankine singularities and Fourier-Kochin distributions of elementary waves over the boundary surface $\Sigma$ and the boundary curve $\Gamma$. The local component $\phi^L$, $\bar{u}^L$ is not considered here.

For the purpose of verifying the foregoing Fourier-Kochin wave representation, the flow due to a source-sink pair is considered here. Fig.1 shows the disturbance velocity $(u,v,w)$ generated by a point source and a point sink, of strength $q = Q/(UL^2) = 0.001$, located at $(x,y,z) = (\pm 0.5, 0, -0.02)$ over the lower half $z \leq 0$ of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2$ with $(a,b,c) = (0.55, 0.05, 0.1)$. The velocity distribution $(u,v,w)$ generated by the point source-sink pair is evaluated, for a Froude number $F = 0.316$, using integral representations of the Green function given in [7]. The upper half of Fig.2 depicts the free-surface elevation, computed using integral representations of the Green function, due to the source-sink pair. The lower half of Fig.2 depicts the free-surface elevation obtained using the Fourier-Kochin wave representation and the velocity distribution generated by the source-sink pair at the ellipsoidal boundary surface depicted in Fig.1. The free-surface elevations computed using expressions for the Green function (upper half) and reconstructed using the Fourier-Kochin wave representation (lower half) are not identical in the vicinity of the ellipsoidal boundary surface because the local-flow component $u^L$ is ignored in the Fourier-Kochin wave representation. The wave elevations shown in Fig.3 along the four longitudinal cuts $y = 0, y = 0.06, y = 0.1, y = 0.5$ show that the local component $u^L$ in fact is only significant in the vicinity of the elliptical boundary curve.

The results depicted in Figs 1-3 provide a verification of the Fourier-Kochin representation of waves. Furthermore, Fig.3 shows that the wave component is dominant even in the nearfield. Illustrative practical applications of the Fourier-Kochin representation of waves are given in [8,9]. Specifically, the Fourier-Kochin representation of steady ship waves is coupled with nearfield calculations based on the Euler equations in [8] and is applied to the design of a wave cancellation multihull ship in [9].

References
[9] C. Yang, F. Noblesse, R. Löhrer, D. Hendrix (2000b) Practical CFD applications to design of a wave cancellation multihull ship, 23rd Symp. on Naval Hydrodyn., Val de Reuil, France
Fig. 1. Velocity distribution generated by source-sink pair at boundary surface

Fig. 2. Wave patterns due to source-sink pair
top: wave pattern computed using Green function
bottom: wave pattern reconstructed using Fourier-Kochin wave representation
Fig. 3. Wave elevations along four cuts at \( y = 0, 0.06, 0.1, 0.5 \) for \( F = 0.316 \)