# Semi analytical solution for heave radiation of the air cushion supported vertical circular cylinder in water of finite depth

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## Introduction

The hydrodynamic problem of air cushion supported floating bodies appears in several situations of marine hydrodynamics : the use of the air cushions was the only way to tow the Gravity Base Structures (GBS) from dry dock to the field of production, the surface effect ships (SES) use the advantages of the air cushions to increase their performances, the succion anchors in the phase of instalation, the projects of huge Mobile Off shore Units (MOB), the projects of the bottomless FPSO-s, ... From the hydrodynamic point of view all these situations are similar. The hydrodynamic behaviour of the structure change considerably, and the air cushion effects should be taken into account properly in the mathematical model. Several numerical methods were proposed [1, 2, 3, 4]. Even if these methods solves the same physical problem, the approaches are quite different. The motivation for this work comes from the necessity to have the analytical solution for one particular boundary value problem (BVP), which may be used for the benchmark purposes. For this objective, we chose the case where the air cushion effects are most pronounced which is the heaving motion of the body. In order to be able to produce semi-analytical solution the case of the vertical circular cylinder in water of finite depth is considered.

## Theory

All definitions correspond to the figure 1. We skip the basic steps, and just recall the final (frequency domain) boundary value problem (BVP) which have to be solved [2]:

$$\left.\begin{array}{ll}
\Delta\psi = 0 & in \;\; \Omega \\
-\nu\psi + \dfrac{\partial\psi}{\partial z} = 0 & z = 0, \; r > a \\
-\nu\psi + \dfrac{\partial\psi}{\partial z} + \dfrac{\alpha}{A_{c0}} \displaystyle\iint_{S_{I0}} \dfrac{\partial\psi}{\partial z} dS = \alpha & z = -d, \; r < a \\
\dfrac{\partial\psi}{\partial r} = 0 & -d < z < 0, \; r = a \\
\dfrac{\partial\psi}{\partial z} = 0 & z = -h \\
\lim[\sqrt{\nu r}(\dfrac{\partial\psi}{\partial r} - i\nu\psi)] = 0 & r \to \infty
\end{array}\right\} \tag{1}$$

where $A_{c0} = 2a\pi$ and $\alpha$ is the nondimensional number accounting for the air cushion effects:

$$\alpha = \kappa \frac{p_{c0} A_{c0}}{\varrho g V_{c0}} \tag{2}$$

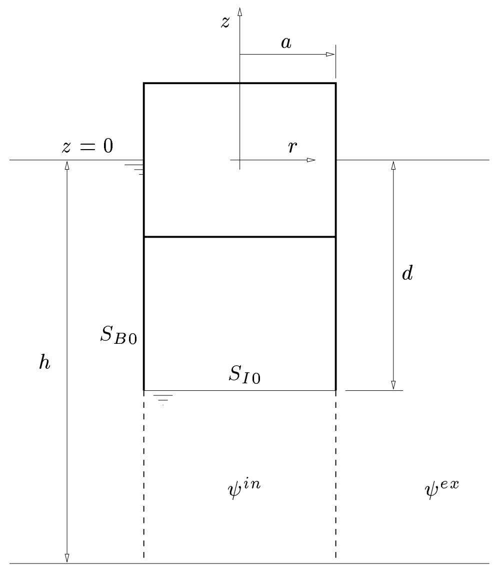



Figure 1: *Basic configuration and notations*

### Potential decomposition

As usually, we make use of the eigenfunction expansions to represent the potential in the different domains (exterior and interior). We note that,due to the fact that only heave is considered, only the $0-th$ Fourier mode in the Fourier series expansion in circumferential direction, need to be considered.
In the exterior domain we write the well known representation:

$$\psi^{ex} = c_0 f_0(z) H_0(k_0 r) + \sum_{n=1}^{\infty} c_n f_n(z) K_0(k_n r) \tag{3}$$

where $\nu = k_0 \tanh k_0 h = -k_n \tan k_n h$, $H_0$ is the Hankel function, $K_0$ is the modified Bessel function of the second kind, and:

$$f_0(z) = \frac{\cosh k_0(z+h)}{\cosh k_0 h} \quad , \quad f_n(z) = \frac{\cos k_n(z+h)}{\cos k_n h} \tag{4}$$

Concerning the interior domain, the situation is little bit more complicated due to the presence of the air cushion interface i.e. the "unusual" boundary condition which should be applied on it. In order to find the correct eigenfunction expansion for the interior potential, first we assume the solution as a combination of one homogeneous and one particular part:

$$\psi^{in} = \psi_h^{in} + C\psi_p^{in} \tag{5}$$

where the associated free surface conditions at interface are:

$$-\nu\psi_h^{in} + \frac{\partial \psi_h^{in}}{\partial z} = 0 \quad , \quad -\nu\psi_p^{in} + \frac{\partial \psi_p^{in}}{\partial z} = 1 \tag{6}$$

For the homogeneous part $\psi_h^{in}$ we can write the representation similar to $\psi^{ex}$ :

$$\psi_h^{in} = b_0 g_0(z) J_0(k_0 r) + \sum_{n=1}^{\infty} b_n g_n(z) I_0(k_n r) \tag{7}$$

where $J_0$ stands for the Bessel function of the first kind, $I_0$ stands for the modified Bessel function of the first kind, and:

$$g_0(z) = \frac{\cosh \kappa_0 (z+h)}{\cosh \kappa_0 (h-d)} \quad , \quad g_n(z) = \frac{\cos \kappa_n (z+h)}{\cos \kappa_n (h-d)} \tag{8}$$

with $\nu = \kappa_0 \tanh \kappa_0 (h-d) = -\kappa_n \tan \kappa_n (h-d)$.
The particular solution may be found, by inspection:

$$\psi_p^{in} = -\frac{1}{\nu} \tag{9}$$

From the boundary condition at the interface, we can now deduce the constant $C$:

$$C = \alpha[1 - \frac{\nu}{2a\pi} \int_0^{2\pi} \int_0^a \psi_h^{in} r dr d\theta] = \alpha\{1 - \nu[b_0 \frac{J_1(\kappa_0 a)}{\kappa_0} + \sum_{n=1}^{\infty} b_n \frac{I_0(\kappa_n r)}{\kappa_n}]\} \tag{10}$$

which gives the following final expression for $\psi^{in}$:

$$\psi^{in} = -\frac{\alpha}{\nu} + b_0[\frac{\alpha}{\kappa_0} J_1(\kappa_0 a) + g_0(z) J_0(\kappa_0 r)] + \sum_{n=1}^{\infty} b_n[\frac{\alpha}{\kappa_n} I_1(\kappa_n a)] + g_n(z) I_0(\kappa_n r)] \tag{11}$$

In order to obtain the unknown coefficients $b_n$ and $c_n$ we make use of the orthogonality of the eigenfunctions by applying the boundary condition on the cylinder and the continuity conditions (potential and radial velocity) at the intersection of the exterior and interior domains:

$$\left.\begin{aligned} \int_{-h}^{-d} \psi^{ex} g_n(z) dz &= \int_{-h}^{-d} \psi^{in} g_n(z) dz \qquad ; \qquad n = 0, \infty \\ \int_{-h}^{0} \frac{\partial \psi^{ex}}{\partial r} f_n(z) dz &= \int_{-h}^{-d} \frac{\partial \psi^{in}}{\partial r} f_n(z) dz \qquad ; \qquad n = 0, \infty \end{aligned}\right\} \tag{12}$$

This leads to the following type of the linear system of equations for $b_n$ and $c_n$:

$$\left.\begin{aligned} b_l &= c_0 D_{l0} + \sum_{n=1}^{\infty} c_n D_{ln} + F_l \qquad ; \qquad l = 0, \infty \\ c_l &= b_0 E_{l0} + \sum_{n=1}^{\infty} b_n E_{ln} + G_l \qquad ; \qquad l = 0, \infty \end{aligned}\right\} \tag{13}$$

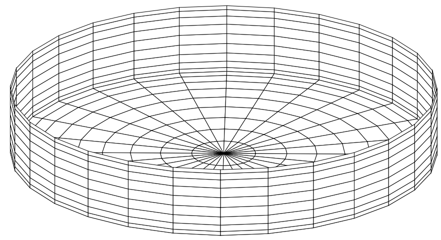
where the coefficients $D_{ln}$, $E_{ln}$, $F_l$ and $G_l$ follow from (12).

## Forces

The forces are obtained by integrating the pressure over the body surface (hydrodynamic and air cushion pressure). It may be shown [2] that, in the case of radiation, the force can be written in the following form:

$$F_{ij} = -\omega^2 A_{ij} + i\omega B_{ij} = \varrho \omega^2 [\iint_{S_{B0}} \varphi_{Rj} N_i dS + \frac{\alpha}{1+\alpha} \iint_{S_{I0}} \varphi_{Rj} N_i dS] \tag{14}$$

where $A_{ij}$ and $B_{ij}$ are the well known added mass and damping coefficients and $\varphi_{Rj}$ is the radiation potential (in our case $\psi$).



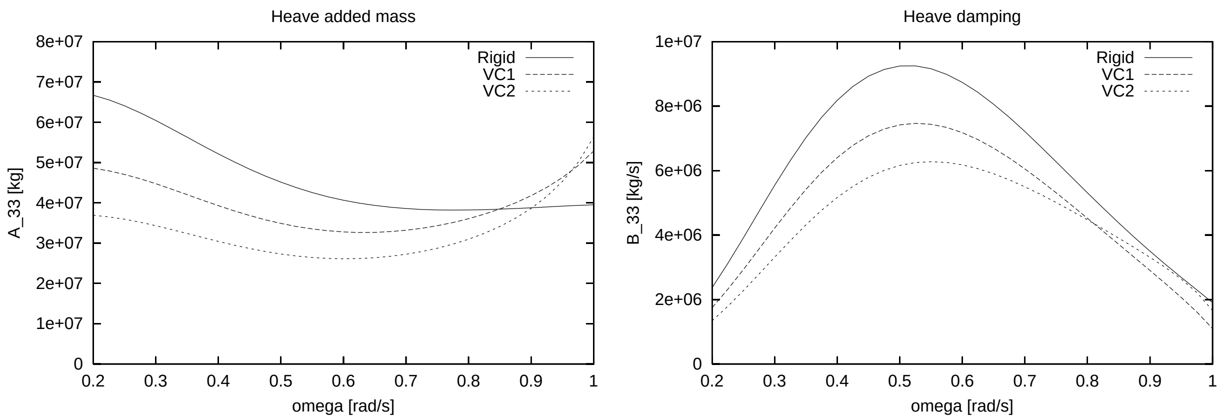



Figure 2: *Added mass and damping.*

### Some results

On the figure 2 we present the numerical results for the heave added mass and damping for the cylinder with the radius of $30m$ and with the draught of $10m$. Static pressure in the air cushion is $p_{c0} = 2 \cdot 10^5 Pa$ and $\kappa = 1.4$. Two values of the static air cushion volume $V_{c0}$ are considered so that the notation VC1 stands for $V_{c0} = 14137.2m^3$ and VC2 for $V_{c0} = 28174.3m^3$. We can clearly appreciate the influence of the air cushion on these coefficients.

## References

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