# THE METACENTRE IN THE STABILITY OF SHIPS. SOME DIFFICULTIES

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## 1 Introduction

The conventional theory of the metacentre is presented in many textbooks, e.g. in [1] and [2]. I have always had difficulties with this piece of theory, which I shall try to explain in the present note. We consider a body floating in equilibrium on the surface of a fluid, the immersed volume being $V$. Let the body be subjected to an external couple, so that it takes up an inclined position ; when the fluid has come to rest it is assumed that the immersed volume is again equal to $V$. If the external couple is now removed, does the floating body tend to return to its equilibrium position ? This is the problem which is treated in textbooks in the following manner.

We take axes $OXYZ$ fixed in the body. It is almost obvious that the possible constrained positions with immersed volume $V$ are defined by two independent angular parameters. The possible positions of the free surface in the system $OXYZ$ are the *Planes of Flotation*, all of which touch the *Surface of Flotation*. In any such inclined position the force system consists of the constraining couple, together with the force of flotation and the (vertical) weight of the body through its centre of gravity. The force of flotation is the resultant of the hydrostatic pressures acting on the immersed part of the body, it is a vertical upward force and is equal and opposite to the weight. As is well known, its line of action (*Line of Buoyancy*) goes through the *Centre of Buoyancy* which is the centroid of the immersed part of the body for that inclination. When the inclination is varied, it is seen (since there are two independent parameters) that the centres of buoyancy form a surface, the *Surface of Buoyancy*, in the coordinate system $OXYZ$ fixed in the body.

The lines of buoyancy form a doubly infinite system of straight lines (i.e a linear congruence) in the system of axes fixed in the body. It can also be shown that the lines of buoyancy are the normals to the surface of buoyancy, see [1], Art.49. For any given inclination we can readily find the relevant line of buoyancy, because it is vertical. So far no mention has been made of the metacentre.

The conventional theory now proceeds as follows: it is assumed that the body is rotated about its longitudinal axis, and that we are interested only in certain small angles of inclination. The corresponding lines of buoyancy then form a one-parameter set, with a cusped envelope. To a close approximation it may now be assumed that these lines of buoyancy pass through the cusp which is given the name of metacentre. Then the relative positions of the cusp and of the centre of gravity determine the direction of the constraining couple, and thus they determine the stability of the body in the inclined positions which we have been considering.

There are some difficulties which will have occurred to many readers of this piece of theory . The first difficulty is concerned with the lines of buoyancy, the remaining difficulties are concerned with the fluid motion.

1. The lines in a one-parameter or two-parameter set of lines do not intersect in general, and therefore do not form an envelope.
2. The calculation refers to the body at rest in its constrained position, whereas stability refers to a body in motion.
3. When the constraint is removed, the body begins to move, and so does the fluid. The inertia of the fluid should enter into the calculation.
4. The stability is not determined at the initial instant, it is the end result of a motion in time. This motion starts when the constraint is removed, and ends when the body and fluid finally come to rest. An additional stability argument is considered in the Discussion below.

5. To follow the motion which has just been described we should have to solve a system of non-linear equations in time. These equations describe the motion of both the body and the fluid.

6. Even if the original inclination is small, the resulting linearized wave motion is not readily treated. (See [3] for a related but simpler calculation which involves the coefficients of virtual mass and moment of inertia and the coefficients of damping for all frequencies.)

These difficulties suggest that we may perhaps hope to find an initial motion. When the constraining couple is removed, how does the body begin to move ?

## 2 A modified problem: To find the initial acceleration

As before, we consider a body floating in equilibrium on the surface of a fluid, the immersed volume being $V$. Let the body be subjected to an external couple, so that it takes up an inclined position; when the fluid has again come to rest it is assumed that the immersed volume is again equal to $V$. If the external couple is now removed, what is the initial linear and angular acceleration of the body ?

Lines of buoyancy, centres of buoyancy, and the surface of buoyancy are defined as above. When the constraining couple is removed the body and the surrounding fluid will immediately be accelerated. We need to know the initial reaction of the fluid on the body. This is determined by solving a boundary-value problem. We take coordinate axes $Oxyz$ fixed in space, such that initially the free surface is at $z = 0$. We denote by $\phi(x, y, z)$ the initial acceleration potential, satisfying Laplace's equation. It can be shown that the boundary condition on the initial free surface $z = 0$ is then $\phi(x, y, 0) = 0$. Thus $\phi(x, y, z)$ can be continued by reflection into the upper half-space by the relation

$$\phi(x, y, z) = -\phi(x, y, -z)$$

and its region of definition is the whole of space, bounded internally by the immersed boundary-surface of the body , and by its reflection in the plane $z = 0$. Together these two surfaces will form a closed body, with a discontinuity of slope along the water-line. To an angular acceleration of the immersed surface about an axis there will correspond an angular acceleration of the reflected surface about the reflected axis, and to a linear acceleration of the immersed body there will correspond a linear acceleration of the reflected body. Thus for any prescribed linear and angular acceleration of the immersed body the potential $\phi$ is well defined in principle outside the closed (immersed plus reflected) body, and the actual linear and angular acceleration can then be found by applying the equations of motion. The boundary conditions on this closed body do not represent a rigid-body motion and therefore the fluid-force system cannot be expressed by the familiar virtual-mass and virtual-moment-of-inertia tensors. However, the acceleration potential can still be found, at least in principle. The calculation involves the determination, from the body contours, of the lines of buoyancy and of the surface of buoyancy, evidently a demanding task which is simplified because the lines of buoyancy are the normals to the surface of buoyancy.

## 3 Discussion and comments

The argument in the two earlier sections has not yet involved any envelope or cusp, and therefore no metacentre. These become involved when we include the hydrostatic forces in the calculation of the initial acceleration. As has been noted, the lines of buoyancy, forming a normal congruence, do not in general intersect each other and therefore cannot form an envelope. We now consider the properties of such a congruence, which are well known. Let us take a point $P$ on the surface of buoyancy. Each plane containing the normal at $P$ intersects the surface of buoyancy in a plane curve, with a well-defined curvature at $P$ which depends on the orientation of the plane. It is shown in textbooks of differential geometry that for one orientation the curvature has a maximum and for another orientation the curvature has a minimum. The corresponding directions on the surface of buoyancy are known as the principal directions of curvature and are orthogonal. To each principal direction there corresponds a centre of curvature, (the normals to the surface pass close to the centre of curvature but do not pass through the centre of curvature) and these centres of curvature lie on a surface of two sheets. Every line of buoyancy is tangent to this surface at each of the two centres of curvature. The lines do not form an envelope, and there are no cusps. It seems possible, however, that for a long ship and for inclinations about the longitudinal axis the lower centre of curvature can be taken as an approximate metacentre, and that the moment of the fluid pressure can be described by a virtual-moment-of-inertia tensor. For a long ship and for an inclination about the longitudinal axis of the ship we see that the lower centre of curvature in the mid-section correponds to the conventional metacentre. In obtaining these considerations use has been made of well-known results in hydrostatics, hydrodynamics and the differential geometry of surfaces, and the assumptions of the modified conventional theory have been clarified, but not its relevance to the original problem of stability.

I found a history of the metacentre in a classical work of scholarship, the Encyclopädie der mathematischen Wissenschaften, published between 1898 and 1920. The most relevant article is by P. Stäckel, [4]. The theory is essentially due to Bouguer [6] who introduced the metacentre, to Euler [5], and to Dupin [7] who introduced lines of buoyancy and surfaces of buoyancy. I believe that Dupin in particular understood very well all the difficulties which I have raised. ( I have not myself consulted these works.) In the middle of the nineteenth century it was pointed out that the static couple cannot by itself tell us anything about stability, since the fluid motion has been ignored. (We may perhaps feel that the static couple and the fluid couple must be in the same direction, more or less, but this has not been proved.) It is not suggested in Stäckel's article that the initial acceleration can be found, or that the linearized motion can be found.

There is an additional stability argument based on energy, which may occur to some readers. We consider the system at the instant when the constraining couple has just been removed. The kinetic energy at this instant vanishes. A wave motion then takes place, and energy is transferred to infinity. After a long time the system is again at rest. Evidently the potential energy of the floating body in the final state must be less than the potential energy in the initial state. Thus for stability the vertical distance between the centre of gravity and the plane of flotation must be less in the inclined position than in the upright position. The surface of buoyancy, and therefore the metacentre, is not involved in this argument.

## References


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[4] Stäckel, P. (1905) Elementare Mechanik. Schwimmende Körper, *Encyclopädie der mathematischen Wissenschaften*, Leipzig, Band 4, Teil 1, Art 6, **39**, 657-658.

[5] Euler, L. (1749) *Scientia Navalis*, St. Petersburg.

[6] Bouguer, P. (1746) *Traité du navire*, Paris.

[7] Dupin, C. (1822) *De la stabilité des corps flottants*, Paris.