DYNAMICS OF THE TRANSIENT LEADING PART OF A WAVE TRAIN

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1 Introduction

The dynamics of waves along the ocean surface determines the input parameters in wave analysis of stationary offshore structures and moving ships, and in the calculation of induced loads in tension legs and risers. Recent experimental investigations at the University of Oslo have shown that the leading waves in a wave group may introduce special loads on the structures. This includes in particular high-frequency loads (ringing). The experimental studies have prompted the present investigation. Questions we have in mind include why leading waves of a wave train can reach appreciable heights? What are the induced velocities and accelerations of the waves? Do moderately steep to steep waves exhibit special features compared with small amplitude waves?

To analyze these questions we derive a very accurate model for a wave train generated by a pneumatic wavemaker.

2 Mathematical formulations

The classical two-dimensional gravity wave equations for a potential flow over a horizontal bottom are

$$\phi_{xx} + \phi_{yy} = 0 \qquad \text{for} \qquad -h \leq y \leq \eta, \tag{1}$$

$$\phi_y = 0 \qquad \text{at} \qquad y = -h, \tag{2}$$

$$\phi_y - \eta_t - \eta_x \phi_x = 0 \qquad \text{at} \qquad y = \eta, \tag{3}$$

$$\tilde{p} + g\eta + \phi_t + \tfrac{1}{2}\phi_x^2 + \tfrac{1}{2}\phi_y^2 = 0 \qquad \text{at} \qquad y = \eta, \tag{4}$$

where ϕ is the velocity potential, h is the mean depth, g is the acceleration due to gravity, η is the surface elevation from rest and $\tilde{p}$ is a forcing pressure at the surface.

To numerically solve the equations, it is convenient to rewrite the system (1)–(4) in a form which involves quantities at the surface only. Using the derivative laws

$$\tilde{\phi}_x = \widetilde{\phi_x} + \eta_x \widetilde{\phi_y}\,, \qquad \tilde{\psi}_x = \widetilde{\psi_x} + \eta_x \widetilde{\psi_y}\,, \qquad \tilde{\phi}_t = \widetilde{\phi_t} + \eta_t \widetilde{\phi_y}\,,$$

(ψ being the stream function, $\psi = 0$ at the bottom, and we denote quantities at the surface by tildes), and the flow conservation law, equations at the surface can be rewritten as:

$$\eta_t + \tilde{\psi}_x = 0, \tag{5}$$

$$\tilde{\phi}_t + g\eta + \frac{1}{2}\,\frac{\tilde{\phi}_x^2 - \tilde{\psi}_x^2 + 2\,\eta_x\,\tilde{\phi}_x\,\tilde{\psi}_x}{1 + \eta_x^2} = -\tilde{p}. \tag{6}$$

These two equations give temporal evolutions of η and $\tilde{\phi}$. To complete the system we need an equation for $\tilde{\psi}$. Using the holomorphy of $\phi + \mathrm{i}\psi$, the equation for $\tilde{\psi}$ is obtained from the Cauchy integral formula. An infinite tank does not exist in the real world and it is not convenient for computations. We then consider a tank of a finite length L, in which $0 \leq x \leq L$. Since we shall use a forcing pressure with an even symmetry, the tank is extended for $x < 0$ by symmetry. Moreover, it is advantageous to periodise the fluid domain to compute solutions with Fast Fourier Transforms. We therefore consider a $2L$-periodic problem. The bottom impermeability is taken into account *via* a Schartzian symmetry. Finally, the equation for $\tilde{\psi}$ is

$$\tilde{\psi}(x,t) = \frac{1}{2L}\,\mathrm{PV}\int_{-L}^{L} \frac{(\,\tilde{\phi}' - \eta'_x\,\tilde{\psi}'\,)\sin\frac{x'-x}{L/\pi} + (\,\tilde{\psi}' + \eta'_x\,\tilde{\phi}'\,)\sinh\frac{\eta'-\eta}{L/\pi}}{\cosh\frac{\eta'-\eta}{L/\pi} - \cos\frac{x'-x}{L/\pi}}\,\mathrm{d}x'$$
$$-\frac{1}{2L}\int_{-L}^{L} \frac{(\,\tilde{\phi}' - \eta'_x\,\tilde{\psi}'\,)\sin\frac{x'-x}{L/\pi} + (\,\tilde{\psi}' + \eta'_x\,\tilde{\phi}'\,)\sinh\frac{2h+\eta'+\eta}{L/\pi}}{\cosh\frac{2h+\eta'+\eta}{L/\pi} - \cos\frac{x'-x}{L/\pi}}\,\mathrm{d}x', \tag{7}$$

where PV is the principal value and $\tilde{\phi}' = \tilde{\phi}(x',t)$, etc. The system (5)–(7) is equivalent to the original one and it involves quantities at the surface only. Moreover, it is purely Eulerian.

To generate from rest a wave of angular frequency σ, we choose a localized pneumatic wavemaker of the form

$$\tilde{p} = gA\,\sin(\sigma t)\,H(t)\,\exp(-x^2 \big/ 2\lambda^2), \tag{8}$$

where H is the Heaviside function, and parameters A and λ are tuned to obtain a wavemaker of maximum efficiency (Wehausen & Laitone 1960).

The analytic solution of the linearized equations is (for $L=\infty$)

$$\eta = \frac{A\lambda\sigma t}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\omega\,e^{-\frac{k^2\lambda^2}{2}}}{\omega+\sigma}\left[\mathrm{sinc}\frac{(\omega-\sigma)t}{2}\cos\frac{(\omega+\sigma)t}{2} - \mathrm{sinc}\,\sigma t\right]e^{\mathrm{i}kx}\,\mathrm{d}k, \tag{9}$$

$$\tilde{\phi} = \frac{-gA\lambda\sigma t}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\omega\,e^{-\frac{k^2\lambda^2}{2}}}{\omega+\sigma}\,\mathrm{sinc}\frac{(\omega-\sigma)t}{2}\sin\frac{(\omega+\sigma)t}{2}\,e^{\mathrm{i}kx}\,\mathrm{d}k, \tag{10}$$

$$\tilde{\psi} = \frac{-\mathrm{i}A\lambda\sigma t}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\omega^3 k^{-1}\,e^{-\frac{k^2\lambda^2}{2}}}{\omega+\sigma}\,\mathrm{sinc}\frac{(\omega-\sigma)t}{2}\sin\frac{(\omega+\sigma)t}{2}\,e^{\mathrm{i}kx}\,\mathrm{d}k, \tag{11}$$

with $\omega^2 = gk\tanh kh$ and $\mathrm{i}^2 = -1$. This solution is used in the numerical scheme.

3 Numerical resolution

Our goal was to obtain a highly accurate approximation, even for long time evolution.

The periodic tank is discretized with a constant spatial step. Derivatives are computed with FFT (pseudo-spectral method). The scheme is hence of infinite order in space and very fast.

The Cauchy integral is discretized with the trapezoidal formula, which is of infinite order for a periodic regular function. This leads to the resolution of an implicit linear system. Calculations are achieved with an optimized SSOR method.

The temporal resolution is carried out in Fourier space. The resolution is very stiff for high frequencies, and thus the time step must be very small. Moreover, the amount of phase error increases with the Fourier wavenumbers and with the length of the time interval. The accuracy condition is more demanding than the stability condition. To avoid this problem, we split equations in linear and nonlinear parts, and we make an analytical integration of the linear part (e.g. (9)–(11)). Only the nonlinear terms remain. Fornberg & Whitham (1978) have used this type of transformation to solve the Korteweg & de Vries equation. The treatment of the linear term is both unconditionally stable and exact. The stability limit and the accuracy are considerably increased. The evolution of nonlinear terms are computed with the fourth-order Runge–Kutta–Gill algorithm.

This scheme is very accurate and relatively fast. It is written in MATLAB and runs on a PC.

4 Preliminary results

We have computed transcient short waves in relative deep water. First we can compare the surface elevations given by the exact numerical solution and the linear solution. In the quasi-steady part of the wave train, the linear and nonlinear solutions are not very different (Fig. 1). This means that a high-order analytical theory can predict correctly the wave field. On the other hand, in the transcient leading part, differences in amplitudes and phases are very important. In the far field where the elevation is small, linear and nonlinear solutions are comparable in both amplitudes and phases. This proves that the significant differences in the transcient leading part are not due to numerical inconsistencies. We observe that the first significant crest focuses energy. It has an oscillating motion, increasing and decreasing alternatively with an amplification until breaking. Such behavior seems to be difficult to describe with classical high-order theories.

We can also consider the horizontal acceleration at the surface (Fig. 2). The back side of the leading crest has an important negative acceleration, which is not the case for the quasi-steady part of the wave train. This phenomenon could have important effects on an obstacle.

Our work is still in progress. Further and more definite results will be presented at the workshop.

References

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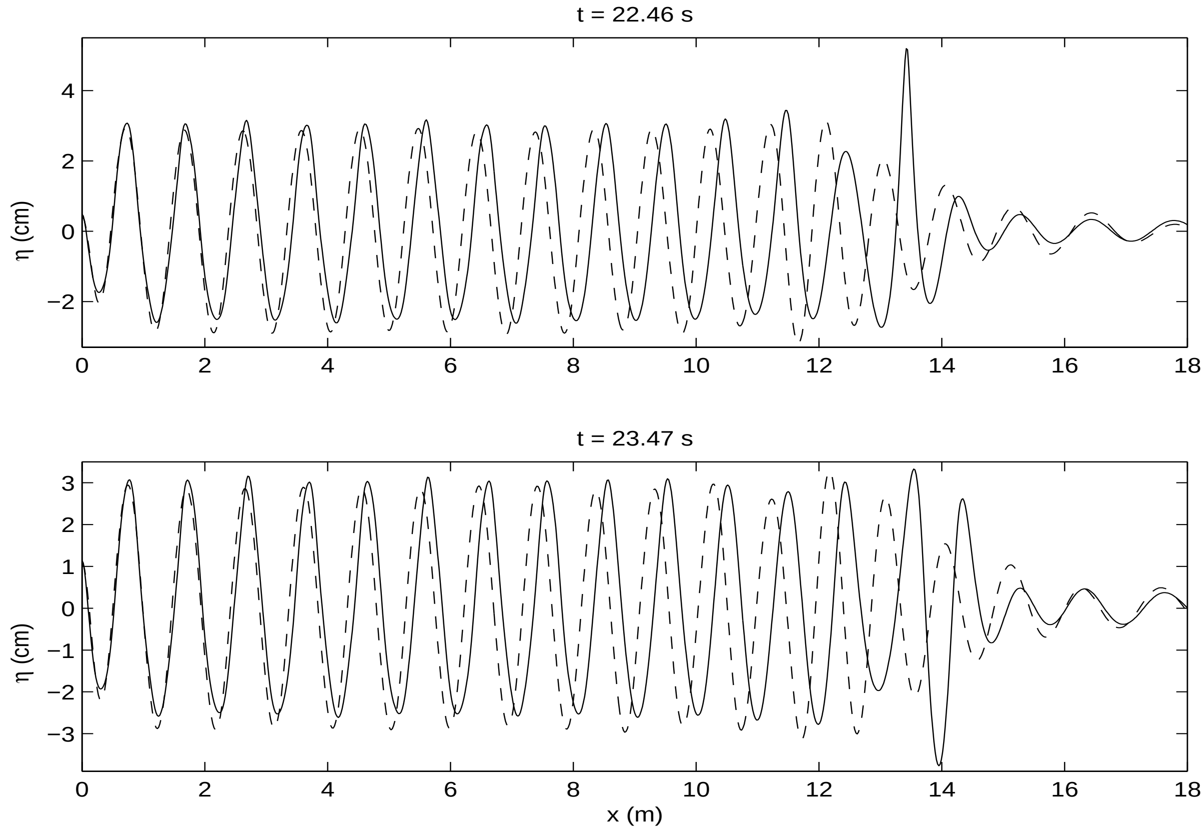

Figure 1: *Surface elevations at two different times.*

(—) exact, (– –) linear, for $g = 9.8\,\mathrm{m\,s^{-2}}$, $h = 0.6\,\mathrm{m}$, $\sigma = 8\,\mathrm{rad\,s^{-1}}$.

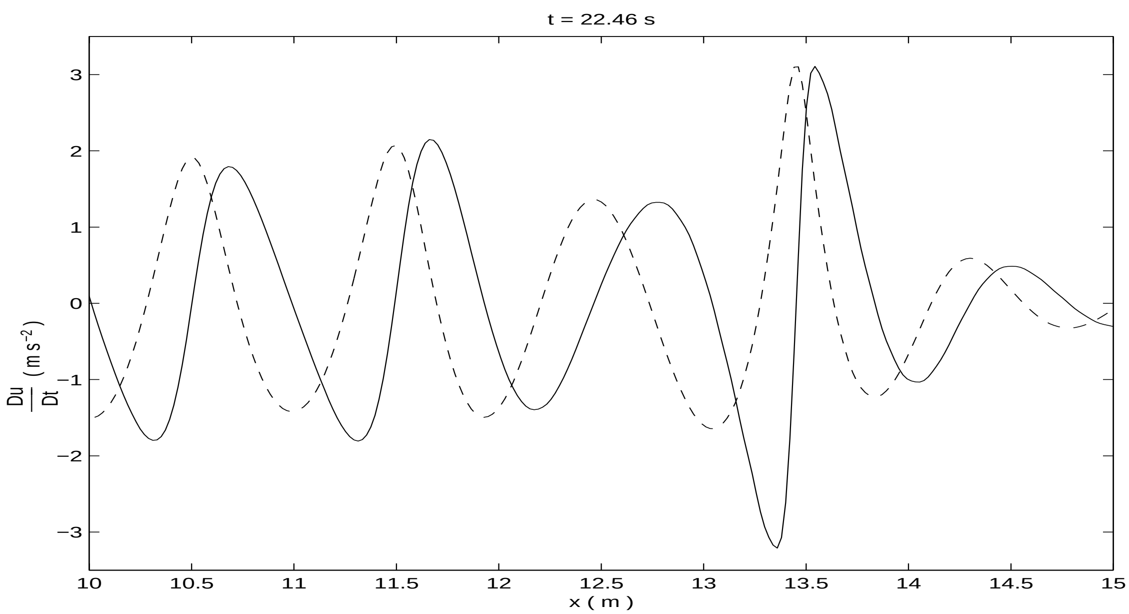

Figure 2: *Horizontal acceleration at the surface.*

(—) acceleration, (– –) surface elevation, for $g = 9.8\,\mathrm{m\,s^{-2}}$, $h = 0.6\,\mathrm{m}$, $\sigma = 8\,\mathrm{rad\,s^{-1}}$.