Hydroelasticity of a buoyant circular plate in shallow water: a closed form solution

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1 Introduction

The number of closed form solutions for dynamic interaction between elastic deformation of a floating thin plate and sea waves is limited. So far, we are aware of Stoker's [1] analytic solution who considered the 2-D case of an infinitely wide elastic thin half-plate floating over shallow water. In this paper we present a closed form solution of the hydroelasticity problem for a thin circular plate floating on the surface of shallow water.

2 Governing equations.

A thin impermeable elastic circular plate of radius r_0 covers a part of the free surface of shallow water of depth h. The elastic plate is swept together with the free surface. The edges of the plate are free of shear forces and bending moments. The amplitude of the incident wave, as well as the free surface elevation induced by the bending and twisting plate are assumed to be small. The fluid is assumed to be inviscid and its motion irrotational. To describe the motion of the plate and the fluid motion the linearized shallow water theory is invoked.

Incident progressive monochromatic wave with wave number k and wave frequency ω propagates in the positive direction of the x-axis. Time-harmonic motion of small amplitude with complex time dependence $\exp(-i\omega t)$ are considered and applied to all first-order oscillatory quantities.

We decompose the physical domain into two regions: *plate* ($r \leq r_0$) and *water* ($r > r_0$). Thus, the velocity potential in the *plate* region is denoted by ϕ_1 and the velocity potential in the *water* region is denoted by $\phi_2^* = \phi_2 + \phi_w$, where ϕ_2 is the velocity potential affected by the plate motion, and ϕ_w is the potential of the ambient wave. Accordingly, the vertical displacement of the plate is denoted by w_1, and the vertical displacement of the free surface w_2^* is represented as a sum of the incident wave elevation w_w and the wave elevation w_2 induced by the plate, i.e.: $w_2^* = w_2 + w_w$.

In the *water* the governing equation for the potential ϕ_2 is:

$$\nabla^2\phi_2 + k^2\phi_2 = 0, \qquad (1)$$

where in the polar system of coordinates (r, θ):

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}. \tag{2}$$

The governing equation for the plate region can be written as:

$$(\nabla^6 + a\nabla^2 + b)\phi_1 = 0, \tag{3}$$

where $a = (\rho g - \omega^2 m)/D$ and $b = \omega^2 \rho / Dh$. Here ρ is the water density, g is the acceleration of gravity, D is the equivalent plate flexural rigidity, and m is the mass of the plate per unit area.

The *water* and *plate* region have to be matched at the boundary $r = r_0$. Generally speaking, at this line the physical quantities such as fluid pressure, free surface and plate elevations, and the fluid velocity components have to be continuous. However, following Stoker [1], we satisfy only two transient conditions: the conservation of energy and mass fluxes through the boundary $r = r_0$ which mathematically are expressed as continuity of the potentials $\phi_{1,2}$ and their first derivatives with respect to r.

Far from the plate the potential ϕ_2 must vanish satisfying the radiation condition. At the free edges of the plate the total force and bending moment vanish. These two requirements give the following dynamic conditions at the boundary $r = r_0$ [2]:

$$\left[\nabla^2 - \frac{(1-\nu)}{r}\left(\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial^2}{\partial \theta^2}\right)\right] w_1 = 0, \tag{4}$$

$$\left[\frac{\partial}{\partial r}\nabla^2 + \frac{1-\nu}{r^2}\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\frac{\partial^2}{\partial \theta^2}\right] w_1 = 0, \tag{5}$$

where ν is the Poisson's ratio.

3 Solution of the boundary value problem.

The governing equation (3) in the plate region can be decomposed as follows:

$$\prod_{m=1}^{3} (\nabla^2 - z_m)\phi_1 = 0,$$

where z_m $(m = 1, 2, 3)$ are the roots of the equation $z^3 + az + b = 0$ which can be found explicitly. Thus, the potential ϕ_1 can be written as $\phi_1 = \sum_{m=1}^{3} \phi_{1,m}$, where each of the potentials $\phi_{1,m}$ satisfies the equation:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} - z_m\right)\phi_{1,m} = 0. \tag{6}$$

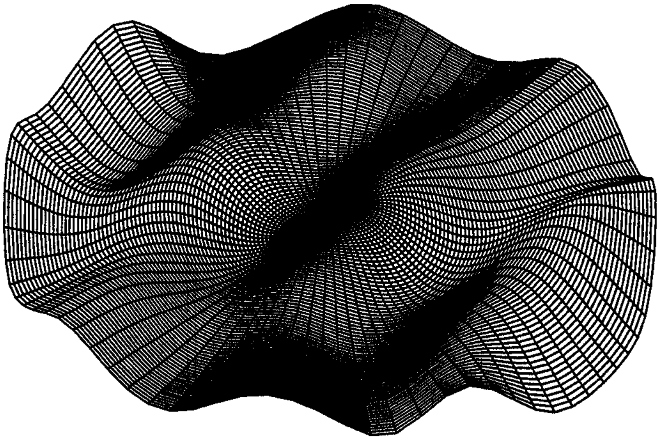

Figure 1: Plate deflection: $-\text{Re}(w_1)$ (plate radius $r_0 = 750$ m, wave amplitude 1 m, wave length 200 m, water depth h=20 m, equivalent flexural rigidity $D = 2 \times 10^{-4} \rho g r_0^4$ Nm, Poisson's ratio $\nu = 0.3$, $\rho = 1025$ kg/m^3, g =9.81 m/s^2). The monochromatic wave propagates from the left upper corner to the right lower corner. In this particular example $\max|\text{Re}(w_1)| \simeq 0.4$.

The sought solution is symmetrical with respect to the x-axis, and, thus, can be represented as a Fourier series $\phi_{1,m} = \sum_{n=0}^{\infty} \varphi_{1,m}^{(n)}(r) \cos n\theta$, where the Fourier coefficients $\varphi_{1,m}^{(n)}(r)$ satisfy the ordinary differential Bessel equation with a bounded solution $\varphi_{1,m} = C_m^{(n)} I_n(\tau_m r)$ ($m = 1, 2, 3$, $n = 1, 2, \cdots$), where $\tau_m = r_0\sqrt{z_m}$. Similarly, the solution of (1) satisfying the radiation condition for each mode n can be represented as $\varphi_2^{(n)} = C_4^{(n)} H_n^{(1)}(kr)$, where $H_n^{(1)}(kr)$ is the Hankel function. Expanding the wave amplitude function and its velocity potential in a Fourier series with respect to the polar angle θ and employing four above mentioned boundary conditions for each $n = 0, 1, \cdots$, we determine four unknown constants $C_m^{(n)}$ ($m = 1, \cdots, 4$). Once these constants are known the quantities of physical interest can be also obtained in a closed form. For example, the plate deflection can be written as:

$$w_1(r, \theta) = -\frac{ih}{\omega} \sum_{n=0}^{\infty} \cos n\theta \sum_{m=1}^{3} \tau_m^2 C_m^{(n)} I_n(\tau_m r). \tag{7}$$

A numerical example is represented in Fig. 1.

References

[1] J.J.Stoker. *Water waves.* Interscience, New York, 1957.

[2] S. Timoshenko, S.Woinowsky-Krieger. *Theory of plates and shells.* McGraw-Hill, New York, 1959.