Hydroelastic Behavior of a Very Large Floating Structure in Waves

Ken Takagi
Department of Naval Architecture and Ocean Engineering
Osaka University, Japan

1. Introduction

A very large floating structure (VLFS) planned in the Mega-Float project in Japan has a thin plate configuration of very large horizontal size, and can be modeled as a membrane sheet of small bending rigidity floating on the water surface. This modeling leads a new free surface condition which governs wave motions in the plate, and makes the analysis of hydroelastic behavior of VLFS simpler. Although this approach has long history in the field of ice floe, at last some studies has applied it to VLFS, for example [1], instead of so-called modal analysis.

Benefit of this approach is not only simplicity of the solution but also making people understand the hydroelastic behavior of VLFS easily, since it insists that the deflection of plate is represented by a wave motion in the plate. In this paper, motion of a corner of VLFS is focused on with this approach because the present plan of VLFS has rectangular corners and these corners could show a singular behavior from a mathematical and/or physical point of view. At first, we start to derive a boundary integral equation with Green's function which satisfies the elastic free surface condition. Then averaging it in vertical direction, we obtain the shallow water approximation of the boundary integral equation. Finally we discuss about several numerical procedures and the behavior of solution i.e. the hydroelastic behavior of VLFS in waves.

2. Boundary Integral Equation

Suppose a flat floating plate of very small draft located in the x-y plane which coincides with the still water surface and z axis is vertically upward. Assuming that the velocity potential Φ is harmonic in time with an angular frequency ω , it can be represented as $\Phi(x,y,z,t) = \Re\left[\phi(x,y,z)e^{i\omega t}\right]$ The velocity potential satisfies Laplace's equation and the zero-draft approximation leads the following boundary conditions.

$$-K\phi + (1+M\nabla^4)\frac{\partial\phi}{\partial z} = 0 \quad \text{at} \quad z=0 \quad \text{in the plate} \tag{1}$$

$$-K\phi + \frac{\partial\phi}{\partial z} = 0 \quad \text{at} \quad z=0 \quad \text{in the fluid region} \tag{2}$$

$$\frac{\partial\phi}{\partial z} = 0 \quad \text{at} \quad z=-h \tag{3}$$

where, $K = \frac{\omega^2}{g}$, $M = D/\rho g$ and $\nabla^2 = \partial_x^2 + \partial_y^2$. D is the flexural rigidity of the plate, g is the gravitational acceleration and ρ is the density of the water. Equation (1) leads a dispersion relation in the plate.

$$K - (1+M\alpha^4)\tanh\alpha h \tag{4}$$

Two roots of (4) are fond on the real axis, innumerable roots are located on the imaginary axis and other four roots are also found in each quarter plane.

Applying Green's second identity to the fluid region covered with the plate and evaluating the integral over the plate by integration by parts, the following boundary integral equation is obtained.

$$\begin{aligned}\phi &= \int_c\int_{-h}^{0}\left(\phi\frac{\partial G_d}{\partial n} - \frac{\partial\phi}{\partial n}G_d\right)dz'dc \\ &+ \frac{M}{K}\int_c\left(\frac{\partial}{\partial z}(\nabla^2\phi)\frac{\partial^2 G_d}{\partial n\partial z} - \frac{\partial^2}{\partial n\partial z}(\nabla^2\phi)\frac{\partial G_d}{\partial z} + \frac{\partial\phi}{\partial z}\frac{\partial^2}{\partial n\partial z}(\nabla^2 G_d) - \frac{\partial^2\phi}{\partial n\partial z}\frac{\partial}{\partial z}(\nabla^2 G_d)\right)|_{z'=0}dc\end{aligned} \tag{5}$$

Where, c is a integral path which coincides with the edge of the plate and n denotes the normal measure inward to the fluid. The Green function satisfies the elastic free surface condition and it is represented as follows:

$$G_d(x,y,z,x',y',z') = \frac{1}{4\pi}\left(\frac{1}{r}+\frac{1}{r_2}\right) - \frac{1}{2\pi}\int_0^\infty \frac{K+(1+Mk^4)k}{K\cosh kh - (1+Mk^4)k\sinh kh}e^{-kh}\cosh k(z+h)\cosh k(z'+h)J_0(kR')dk \tag{6}$$

Where,

$$r^2 = (x-x')^2 + (y-y')^2 + (z-z')^2 \tag{7}$$
$$r_2^2 = (x-x')^2 + (y-y')^2 + (2h+z+z')^2 \tag{8}$$
$$R'^2 = (x-x')^2 + (y-y')^2 \tag{9}$$

Applying the contour integral to the Green function, it is represented by a summation of the Hankel functions.

$$G_d = \frac{i}{2h}\sum_n \frac{H_0^{(2)}(\alpha_n)\cosh\alpha_n(z+h)\cosh\alpha_n(z'+h)}{1+\dfrac{1+5M\alpha_n^4}{2(1+M\alpha_n^4)\alpha_n h}\sinh 2\alpha_n h} - \frac{i}{2h}\frac{H_0^{(1)}(\alpha_1)\cosh\alpha_1(z+h)\cosh\alpha_1(z'+h)}{1+\dfrac{1+5M\alpha_1^4}{2(1+M\alpha_1^4)\alpha_1 h}\sinh 2\alpha_1 h}$$
$$(n = 0,4,5,6,\cdots,\infty) \tag{10}$$

At the edge of the plate, the fact that the moment and the sheering force are free leads following two conditions.

$$\frac{\partial^3\phi}{\partial^2 n\partial z} + \nu\frac{\partial^3\phi}{\partial^2 s\partial z} = 0, \qquad \nabla^2\frac{\partial\phi}{\partial z} + (1+\nu)\frac{\partial^4\phi}{\partial^2\partial n\partial z} = 0 \quad \text{at} \quad z = 0 \tag{11}$$

If we solve the boundary integral equation (5) together with the edge boundary condition (11) and certain forcing term, the deflection of the plate would be obtained.

3. Shallow Water Approximation

It is well known that the water wave problem is greatly simplified by the shallow water approximation, since all evanescent terms are vanished.

Employing the shallow water approximation, the velocity potential can be represented as follows:

$$\phi(x,y,z) = \varphi(x,y) - \frac{1}{2}\nabla^2\varphi(z+h_l)^2 \tag{12}$$

Substituting (12) into (5) and performing integration respect to z', the boundary integral equation (5) becomes very simple.

$$\begin{aligned}\varphi &= -\int_c\left(\varphi\frac{\partial}{\partial n} - \frac{\partial\varphi}{\partial n}\right)G(x,x',y,y')dc \\ &- M\int_c\left(\nabla^2\varphi\frac{\partial}{\partial n} - \frac{\partial}{\partial n}(\nabla^2\varphi)\right)G_1(x,x',y,y')dc - M\int_c\left(\nabla^4\varphi\frac{\partial}{\partial n} - \frac{\partial}{\partial n}(\nabla^4\varphi)\right)G_2(x,x',y,y')dc\end{aligned} \tag{13}$$

where,

$$G(x,x',y,y') = \frac{1}{2\pi}\int_0^\infty \frac{1}{k}[(\left(\frac{K}{\Omega(k)}-1\right)J_0(kR')dk = -i\frac{KH_0^{(2)}(\alpha_0 R')}{2\alpha_0\Omega'(\alpha_0)} + \Im\left[\frac{KH_0^{(1)}(\alpha_1 R')}{\alpha_1\Omega'(\alpha_1)}\right] \tag{14}$$

$$G_1(x,x',y,y') = \frac{1}{2\pi}\int_0^\infty \frac{k^3 h}{\Omega(k)}J_0(kR')dk = -i\frac{\alpha_0^3 hH_0^{(2)}(\alpha_0 R')}{2\Omega'(\alpha_0)} + \Im\left[\frac{\alpha_1^3 hH_0^{(1)}(\alpha_1 R')}{\Omega'(\alpha_1)}\right] \tag{15}$$

$$G_2(x,x',y,y') = \frac{1}{2\pi}\int_0^\infty \frac{kh}{\Omega(k)}J_0(kR')dk = -i\frac{\alpha_0 hH_0^{(2)}(\alpha_0 R')}{2\Omega'(\alpha_0)} + \Im\left[\frac{\alpha_1 hH_0^{(1)}(\alpha_1 R')}{\Omega'(\alpha_1)}\right] \tag{16}$$

Where, $\Omega(k)$ denotes the dispersion relation which is represented as follows:

$$\Omega(k) = K - (1+Mk^4)k^2 h \tag{17}$$

and α_n $(n = 0,1,2,3,4)$ are roots of $\Omega(k) = 0$.
It is noted that higher order terms proportional to $O(h^2)$ are neglected in the integral equation (13). Therefore all equations are consistent up to the order of h, and the two-dimensional velocity potential φ satisfies the free surface condition.

$$K\varphi + (1+M\nabla^4)\nabla^2\varphi h = 0 \tag{18}$$

The Green functions (14), (15) and (16) implies that the velocity potential φ is represented in the eigen function expansion form which are composed of the Hankel functions or the Bessel functions.

4. Corner Problem

Suppose a semi-infinite quarter plate which occupies $(x > 0, y > 0)$ and plane progressive waves coming from the fluid region whose angle of direction is χ. It is obvious from the radiation condition that the contribution of infinity boundary vanishes. Therefore, the path of integral in (13) becomes $x = 0$ and $y = 0$.

$$\begin{aligned}\varphi(x,y) = & - \int_0^\infty \Big[\left(\varphi\frac{\partial}{\partial y'} - \frac{\partial\varphi}{\partial y'}\right) G(x,x',y,0) - M\left(\nabla^2\varphi\frac{\partial}{\partial y'} - \frac{\partial}{\partial y'}(\nabla^2\varphi)\right) G_1(x,x',y,0) \\ & + M\left(\nabla^4\varphi\frac{\partial}{\partial y'} - \frac{\partial}{\partial y'}(\nabla^4\varphi)\right) G_2(x,x',y,0)\Big] dx' \\ & - \int_0^\infty \Big[\left(\varphi\frac{\partial}{\partial x'} - \frac{\partial\varphi}{\partial x'}\right) G(x,0,y,y') - M\left(\nabla^2\varphi\frac{\partial}{\partial x'} - \frac{\partial}{\partial x'}(\nabla^2\varphi)\right) G_1(x,0,y,y') \\ & + M\left(\nabla^4\varphi\frac{\partial}{\partial x'} - \frac{\partial}{\partial x'}(\nabla^4\varphi)\right) G_2(x,0,y,y')\Big] dy' \end{aligned} \tag{19}$$

Similarly a boundary integral equation for the out side of the plate is obtained.

$$\varphi = -\int_0^\infty \left(\varphi\frac{\partial}{\partial y'} - \frac{\partial\varphi}{\partial y'}\right) G_w(x,x',y,0)dx' - \int_0^\infty \left(\varphi\frac{\partial}{\partial x'} - \frac{\partial\varphi}{\partial x'}\right) G_w(x,0,y,y')dy' \tag{20}$$

where,

$$G_w(x,x',y,y') = \frac{1}{2\pi}\int_0^\infty \frac{1}{k}\left(\frac{K}{K-k^2h} - 1\right) J_0(kR')dk = \frac{i}{2}H_0^{(2)}(k_0R') \tag{21}$$

k_0 denotes the root of shallow water dispersion relation.

In order to derive the eigen function expansion of φ, we apply the additional theorem of the Bessel function to decompose the Green functions in (19). But, it is immediately found that the uniform expansion can not be obtained. This result is not surprising because when the observation point is far from the corner point the solution must be represented with a series of the Hankel function, on the contrary, when the observation point is close to the corner the solution has no singularity and can be represented with a series of the Bessel function. It seems this difficulty prevent to find the solution in an eigen expansion form. Therefore we try to solve the boundary integral equation (19) directly.

A difficult point of solving (19) is that the boundary has infinite length. But, fortunately we know the solution far from the corner which coincides with the solution of the semi-infinite half plate problem. It is obvious that this solution is sinusoidal along the boundary. Thus, we assume the sinusoidal distribution of normal velocity along the x-axis as the first step.

$$\begin{aligned} I &= \lim_{\epsilon\to 0}\int_0^\infty e^{-i(k_x - i\epsilon)x'} G(x,x',y,0)dx' \\ &= \lim_{\epsilon,\iota\to 0}\frac{1}{2\pi}\int_0^\infty e^{-i(k_x-i\epsilon)x'}\int_0^\infty \frac{1}{k}\left(\frac{K}{(K-i\iota)\cosh kh - (1+Mk^4)k\sinh kh} - 1\right) J_0(kR')dkdx' \end{aligned} \tag{22}$$

Where, ϵ and ι ensure the radiation condition.

Applying a contour integral to (22), a component of plane waves can be derived.

$$\begin{aligned} I = & -\frac{i}{2\pi}\frac{K}{\alpha_0\Omega'(\alpha_0)}\int_{-\infty}^\infty \frac{e^{-i\alpha_0 R\cosh\theta}}{k_x - \alpha_0\cos(\beta+i\theta)}d\theta - \frac{i}{2\pi}\frac{K}{\alpha_1\Omega'(\alpha_1)}\int_{-\infty}^\infty \frac{e^{i\alpha_1 R\cosh\theta}}{k_x + \alpha_1\cos(\beta+i\theta)}d\theta \\ & - \frac{i}{2\pi}\frac{K}{\alpha_4\Omega'(\alpha_4)}\int_{-\infty}^\infty \frac{e^{-i\alpha_4 R\cosh\theta}}{k_x - \alpha_4\cos(\beta+i\theta)}d\theta + C_{I1} + C_{I2} + C_{I3} \end{aligned} \tag{23}$$

where,

$$C_{I1} = \begin{cases} 0 & \text{when } \frac{\pi}{2} > \beta > \mu_{x0} \\ -i\dfrac{Ke^{-iR\alpha_0\cos(\mu_{x0}-\beta)}}{\alpha_0^2\Omega'(\alpha_0)\sin\mu_{x0}} & \text{when } 0 < \beta < \mu_{x0} \end{cases} \tag{24}$$

$$C_{I2} = -i\frac{Ke^{-iR\alpha_1\cos(\mu_{x1}-\beta)}}{\alpha_1^2\Omega'(\alpha_1)\sin\mu_{x1}} \tag{25}$$

$$C_{I3} = \begin{cases} 0 & \text{when } \beta > \Re[\mu_{x4}] \\ -i\dfrac{Ke^{-iR\alpha_4\cos(\mu_{x4}-\beta)}}{\alpha_4^2\Omega'(\alpha_4)\sin\mu_{x4}} & \text{when } \beta < \Re[\mu_{x4}] \end{cases} \tag{26}$$

$$\frac{k_y}{k_x} = \alpha_n \tan \mu_{xn} \tag{27}$$

$$\left.\begin{matrix} x \\ y \end{matrix}\right\} = R \left\{\begin{matrix} \cos\beta \\ \sin\beta \end{matrix}\right. \tag{28}$$

It is noted that C_{I1} denotes plane progressive waves and it does not affect the other edge i.e. on the line $x = 0$. C_{I2} and C_{I3} are also plane progressive waves, however their wave number is a complex number and these waves decay quickly as the coordinate y becomes large. The first three terms in (23) represent the end effect and the first term is asymptotically proportional to $1/\sqrt{R}$. When the observation point is far from the corner, the end effect vanishes and the solution coincides with the solution of the semi-infinite half plate problem. Similar result is also obtained in the fluid region.

$$W = \lim_{\epsilon \to 0} \frac{1}{2\pi} \int_0^\infty e^{-i(k_x - i\epsilon)x'} G(x, x', y, 0) dx' = -\frac{i}{4\pi} \int_{-\infty}^{\infty} \frac{e^{-ik_0 R \cosh\theta}}{k_x - k_0 \cos(\beta + i\theta)} d\theta + C_{W1} \tag{29}$$

where,

$$C_{W1} = \begin{cases} 0 & \text{when } -\frac{\pi}{2} < \beta < \chi_{x0} \\ -i\dfrac{e^{-iRk_0 \cos(\chi_{x0} - \beta)}}{2k_0 \sin \chi_{x0}} & \text{when } 0 > \beta > \chi_{x0} \end{cases} \tag{30}$$

$$-\frac{k_y}{k_x} = k_0 \tan \chi_{x0} \tag{31}$$

Now, we know the solution at the infinity and the solution decays proportionally to $1/\sqrt{R}$, and the conventional numerical procedure can be applied to get the corner effect. However, the convergence of $1/\sqrt{R}$ is not rapid. Therefore, in order to subtract the contribution of the terms which are proportional to $1/\sqrt{R}$, the following integral is performed.

$$\begin{aligned} I_s &= \int_0^\infty \frac{1}{\sqrt{x'}} e^{-i\alpha_0 x'} G(x, y, x', 0) dx' \\ &= \frac{K(1+i)}{2\sqrt{2\pi\alpha_0}\Omega'(\alpha_0)} \int_{-\infty}^{\infty} \frac{e^{-i\alpha_0 R\cosh\theta}}{\sqrt{\alpha_0 - \alpha_0 \cos(\mu + i\theta)}} d\theta + \frac{K(1+i)}{2\sqrt{2\pi\alpha_1}\Omega'(\alpha_1)} \int_{-\infty}^{\infty} \frac{e^{i\alpha_1 R\cosh\theta}}{\sqrt{\alpha_0 + \alpha_1 \cos(\mu + i\theta)}} d\theta \\ &+ \frac{K(1+i)}{2\sqrt{2\pi\alpha_4}\Omega'(\alpha_4)} \int_{-\infty}^{\infty} \frac{e^{-i\alpha_4 R\cosh\theta}}{\sqrt{\alpha_0 - \alpha_4 \cos(\mu + i\theta)}} d\theta - \frac{K(1+i)}{\sqrt{2\pi\alpha_1}\Omega'(\alpha_1)} \int_{\alpha_0}^{\infty} \frac{e^{-ixs + iy\sqrt{\alpha_1^2 - s^2}}}{\sqrt{s - \alpha_0}\sqrt{\alpha_1^2 - s^2}} ds \\ &+ \frac{K(1+i)}{\sqrt{2\pi\alpha_4}\Omega'(\alpha_4)} \int_{\alpha_0}^{\alpha_0'} \frac{e^{-ixs - iy\sqrt{\alpha_4^2 - s^2}}}{\sqrt{s - \alpha_0}\sqrt{\alpha_4^2 - s^2}} ds \end{aligned} \tag{32}$$

Where α_0' is determined from the geometrical relation between α_0 and α_4 in the complex plane.

It is obvious that $1/\sqrt{x}$ distribution leads only a three-dimensional wave term, however it does not satisfies Green's theorem because of a contribution from the terms which decay more rapid than $1/\sqrt{R}$. Similar results are also obtained in the fluid region.

Utilizing these integrals, the plane wave term and the three-dimensional wave term can be subtracted from the boundary integral equation (19). Then, only the local term which decays rapidly is computed by a conventional numerical procedure.

Several attempts to solve this integral equation will be shown in the workshop and the hydroelastic behavior of VLFS also will be discussed.

Acknowledgement

This work has been done while the author was staying in University of Bristol. The author would very much like to thank Professor David Evans for his valuable discussions and all people in Bristol for their hospitalities.

Reference

[1] Takagi,K. "Water Waves Beneath a Floating Elastic Plate", 13th wwwfb, 1998.