# Rayleigh-Bloch surface waves in the presence of infinite periodic rows of cylinders and their connection with trapped modes in channels

R. Porter & D.V. Evans
School of Mathematics, University of Bristol, Bristol, BS8 1TW, UK

## 1 Introduction

In the present paper we examine the problem of Rayleigh-Bloch guided or surface waves supported by infinite periodic arrays of vertical cylinders extending through the depth in a ocean of constant depth. Rayleigh-Bloch surface waves are described by a localised oscillation contained within the vicinity of the cylinders and decaying exponentially away from the array. In general, Rayleigh-Bloch surface waves propagate energy along the array, though certain parameter choices can also give rise to standing modes as we shall see.

The geometry consists of an infinite array of vertical cylinders which are arranged periodically in the $y$-direction with period $2d$. Within a typical strip of width $2d$, the arrangement of any cylinders is assumed for the time being to be arbitrary, although it will be shown later that it appears that Rayleigh-Bloch surface waves are in general only supported by geometries which are symmetric about a plane $y = constant$.

Rayleigh-Bloch surface waves are characterised by a dominant wavenumber, $\beta$, representing a change in phase of $e^{2i\beta d}$ in the field between corresponding points in adjacent strips. Simple arguments show that when the wavenumber of motion, $k$, is less than the Rayleigh-Bloch wavenumber $\beta$, there can be no radiation away from the cylinders to infinity and the wave is trapped. Associated with the value $k = \beta$ is a frequency $\omega_c$ called the cut-off frequency and $k < \beta$ therefore corresponds to frequencies below the cut-off frequency. Note that there are examples of trapping modes *above* the cut-off frequency (see Evans & Porter [1]), but here we prefer to concentrate on the regime below the cut-off where no radiation to infinity is assured.

There are two aspects to the current work. The first is based on a single periodic row of cylinders having *arbitrary* cross-section and in the second part we consider the possibility Rayleigh-Bloch modes in the presence of *multiple* periodic rows of *circular* cross-section. The first of these was discussed in some detail at last years Workshop (Porter & Evans [2]) and will shortly be published in Porter & Evans [3]. However, in trying to explain the presence of near-trapping by a finite array of cylinders using the results from a Rayleigh-Bloch approach it was noted that our figures calculated for the near-trapped mode wavenumber were identical to those found by Utsunomiya & Eatock-Taylor [4] who had computed the trapped modes in a uniform width channel having either Neumann or Dirichlet conditions imposed upon them and spanned by an arbitrary number of cylinders. This is no coincidence, and we explain in §4 how the Rayleigh-Bloch approach applied to a single cylinder can be used to generate the wavenumbers and corresponding solutions for the Neumann and Dirichlet trapped modes about any number of cylinders spanning a channel by choosing appropriate values of the Rayleigh-Bloch wavenumber $\beta$. This mechanism also allows one simply to count and categorise the number and type of trapped modes in a given situation.

Whereas a Green function method must be applied to the case of cylinders with arbitrary cross-section to yield a first-kind integral equation for the unknown potential, described briefly in §3, in §5 we examine the case of multiple periodic rows of cylinders having *circular* cross-section. This allows an extension of Evans & Porter [5] who considered trapped modes about multiple cylinders lying on the centre of a Neumann or Dirichlet channel to Rayleigh-Bloch waves using a combination of multipole methods and Graf's addition theorem for Bessel functions. On applying these techniques, the Rayleigh-Bloch solutions turn out to correspond to the solution to an infinite homogeneous linear system of equations. Although algebraically quite complicated, there are no particular difficulties in computing the approximate solutions of the truncated system which converge very rapidly. Significantly, as long as the cylinders are placed symmetrically such that their axes lie in the same plane $y = constant$, the system of equations turn out to be *real* and the corresponding determinant is therefore also real. Since we are seeking real values of wavenumber $k < \beta$ for a given $\beta$ for which the determinant vanishes, the realness of the determinant is a desirable property to have. As in the previous treatment of a single row of arbitrary cylinders, the trapped modes about an array of $M \times N$ circular cylinders occupying a Neumann or Dirichlet channel can be recovered from the Rayleigh-Bloch solutions for $N$ rows of cylinders by choosing the appropriate values of $\beta$. These results may have an impact on proposed designs for offshore airports supported by large rectangular arrays of vertical columns as well as in other areas, such as the design of heat exchangers consisting of tube bundles arranged within a waveguide.

## 2 Rayleigh-Bloch formulation

Consider an infinite periodic linear array of cylinders each of arbitrary cross section, having boundary $\partial D$, and uniform throughout the depth. The generators of the cylinders are aligned with the depth coordinate, $z$, and positioned at $(x, y) = (0, 2jd)$, $j = 0, \pm1, \pm2, \ldots$. Linearised potential theory states that there exists a velocity potential $\Phi(x, y, z, t) = \mathrm{Re}\{\phi(x, y) \cosh k(z - H)\mathrm{e}^{\mathrm{i}\omega t}\}$ where $H$ is the depth, the motion is assumed harmonic in time $t$ with angular frequency $\omega$ and the wavenumber $k$ is related to $\omega$ and gravity, $g$, by the dispersion relation $\omega^2/g = k \tanh kH$. The two-dimensional complex velocity potential now satisfies the Helmholtz equation,

$$\phi_{xx} + \phi_{yy} + k^2\phi = 0$$

everywhere in the field apart from on the boundaries of the cylinders where

$$\phi_n = 0,$$

and $n$ denotes the normal derivative with respect to the cylinder surface. Because the geometry has periodicity of $2d$ in the $y$-direction, we may relate the potential through

$$\phi(x, y + 2jd) = \mathrm{e}^{\mathrm{i}\beta 2dj}\phi(x, y), \qquad -\infty < j < \infty \tag{1}$$

which expresses simply that there is a change in phase of $\mathrm{e}^{\mathrm{i}\beta 2d}$ from the field point at $y$ to the field point at $y + 2d$ in the adjacent strip. Thus the total field can be obtained by referring to a single strip of width $2d$ containing the cylinder. We therefore restrict our attention to the strip $S_1 \in \{-\infty < x < \infty,\ |y| \leq d\}$ and impose appropriate periodicity conditions on the lines $y = \pm d$ of

$$\phi(x, d) = \mathrm{e}^{\mathrm{i}\beta 2d}\phi(x, -d), \qquad \phi_y(x, d) = \mathrm{e}^{\mathrm{i}\beta 2d}\phi_y(x, -d), \tag{2}$$

with (1) providing the extension to all $(x, y)$.

## 3 A single periodic row of arbitrary cross-section cylinders

In the case of a single row of cylinders having *arbitrary* cross section $\partial D$, we take a Helmholtz Green function, $G$, for periodic domains (see, for example, Linton [6]) and use it in conjunction with $\phi$ in Greens Identity applied to the strip $S_1$ to yield the integral equation

$$\int_{\partial D} \phi(p)\frac{\partial G(p|q)}{\partial n_q}\,\mathrm{d}s_q = \begin{cases} \frac{1}{2}\phi(p), & p \in \partial D, \\ \phi(p), & p \notin \partial D. \end{cases} \tag{3}$$

where, here, $p$ and $q$ are labels for the points $(x, y)$ and $(\xi, \eta)$ respectively and $n_q$, $s_q$ represent normal derivatives and arclengths. This integral equation can be solved numerically by, for example discretisation and collocation as described in Porter & Evans (1999) resulting in a discrete complex system of equations to be satisfied. It can be shown that there are two circumstances under which the corresponding complex determinant can be made real: if the cylinder (and hence $\partial D$) is symmetric about a plane of $y = constant$; or if the Green function itself is real which happens for the particular value of $\beta = \pi/2d$. Some examples of such results will be presented.

# 4 Construction of trapped modes about $M$ symmetric cylinders spanning a channel

We assume that each cylinder cross-section, $\partial D$ is symmetric. First we redefine the location of the origin, and place it for convenience to coincide with a line of geometric symmetry between bisecting two adjacent cylinders in the infinite array. Then $y = 2Md$ is a corresponding line of symmetry bisecting the $M$th and $(M+1)$th cylinders and the strip $S_M$ of width $2Md$ contains $M$ cylinders. It can be shown using either physical arguments, or directly from (3) assuming symmetry that

$$\phi(x,y) = \mathrm{e}^{-2\mathrm{i}\alpha}\bar{\phi}(x,-y), \qquad \alpha \in \mathbb{R}.$$

Let us consider the $M+1$ particular values of $\beta$ given by

$$\beta d = n\pi/2M, \qquad n = 1, \ldots M; 2M \tag{4}$$

giving rise to $M+1$ distinct values of $kd(\beta d)$ which, used in (1), show that

$$\left.\begin{aligned} \phi(x, y+2Md) &= (-1)^n \phi(x,y) \\ \phi_y(x, y+2Md) &= (-1)^n \phi_y(x,y) \end{aligned}\right\}, \qquad n = 1, \ldots, M; 2M$$

such that odd (even) values of $n$ correspond to ***half-periodic*** (*periodic*) potentials in each strip $S_M$. We may, without loss of generality, incorporate an arbitrary phase into the potential $\phi$. Thus, we consider the potential

$$\chi(x,y) = \mathrm{e}^{\mathrm{i}\alpha}\phi(x,y) \qquad \text{such that} \qquad \chi(x,y) = \bar{\chi}(x,-y).$$

Now since $\chi(x,y)$ represents Rayleigh-Bloch surface wave solution, then its symmetric and antisymmetric parts, decomposed about the line of geometric symmetry, also satisfy all the conditions of the problem. We can therefore consider the following two potentials

$$\chi^N(x,y) = \mathrm{Re}\{\chi(x,y)\}, \qquad \chi^D(x,y) = \mathrm{Im}\{\chi(x,y)\}$$

such that

$$\chi^N(x,y) = \chi^N(x,-y), \qquad \text{and} \qquad \chi^D(x,y) = -\chi^D(x,-y).$$

and the superscripts $N$ and $D$ denote modes satisfying Neumann and Dirichlet conditions respectively on $y = 0$ and $y = 2Md$. Note that these two potential are purely real, a property which is typical of trapped modes in channels. The particular values of $\beta d = \frac{1}{2}\pi$ and $\beta = \pi$ chosen by taking $n = M$ and $2M$ in (4) are easily shown to correspond to the Neumann ($\chi^N$) and Dirichlet ($\chi^D$) trapped modes respectively about a *single* cylinder in a channel, the solution in the strip $S_M$ being constructed by 'gluing' together $M$ such channels. Note that there is no non-trivial $\chi^D$ ($\chi^N$) solution for $\beta d = \frac{1}{2}\pi$ (*pi*). For the remaining $M-1$ values of $\beta d$ represented in (4), both solutions $\chi^N$, $\chi^D$ are non-trivial and therefore give rise to Neumann *and* Dirichlet modes *for the same value of kd*. Thus, in general, one can state that for $M$ cylinder spanning a channel, there are at least $M$ Neumann trapped modes and at most $M$ Dirichlet trapped modes.

# 5 Multiple periodic rows of circular cylinders

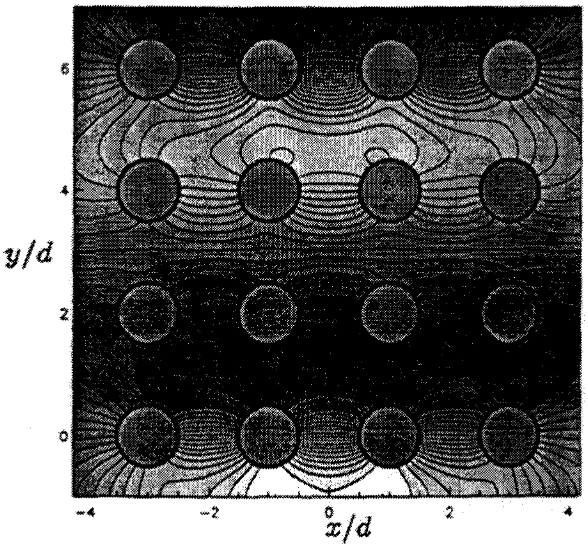
Combining the techniques of multipole potentials and Graf's addition theorem allows one to consider the case of multiple periodically-arranged infinite arrays of cylinders. As described in §2, we reduce the infinite array to the strip $S_1$, now containing $N$ cylinders of radius $a_i$ centred at $(x,y) = (x_i, 0)$,

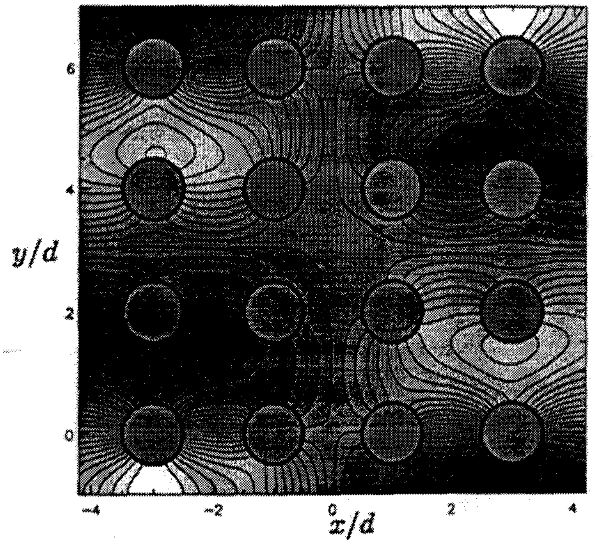
$i = 1, \ldots, N$. The basic building blocks are the multipole potentials $\psi_n(r_i, \theta_i)$ $n = 0, 1, \ldots$, singular at $r_i = 0$ where $r_i^2 = (x - x_i)^2 + y^2$, $\theta_i = \tan^{-1}(y/x - x_i)$, and satisfying the periodicity conditions (2) on $y = \pm d$. Such multipole potentials are derived in Linton & Evans [7]. Thus we write the potential as a sum over all cylinders of the Fourier-type sum of all multipole potentials:

$$\phi(x, y) = \sum_{j=1}^{N} \sum_{n=0}^{\infty} a_n^j \psi_n(r_j, \theta_j).$$

Provided $k < \beta$, the solution is *real* and decays to zero as $|x| \to \infty$. The remaining condition to be satisfied is that of no-flow through the surface of the $N$ cylinders. This requires that the potential to be expressed in terms of the local polar coordinates of cylinder $i$, say, and for this Graf's addition theorem is used since the multipole themselves are expressed as an expansion in terms of Bessel functions. The result of applying the cylinder no-flow condition to the $N$ cylinders is an infinite system of *real* equations, the vanishing of whose corresponding determinant provides the Rayleigh-Bloch solution $kd(\beta d)$.

As in the previous section the trapped modes about a rectangular array of $M \times N$ cylinders can be recovered from the Rayleigh-Bloch solution along $N$ periodic rows of cylinders by choosing the $M + 1$ values of $\beta d$ given in (4). As an example of such modes the two figures below illustrate the free surface amplitudes of the Neumann trapped modes about an array of $4 \times 4$ identical cylinders in a channel (with $\beta d = \frac{3}{8}\pi$). The left figure is the first mode symmetric about $x = 0$ and the figure on the right is the first antisymmetric mode. Further such results will be presented at the Workshop.

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