Uniqueness and trapped modes for a symmetric structure

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Introduction

At the previous workshop McIver (1998) proved that the two-dimensional boundary value problem for linear water waves in the presence of an arbitrary sea-bed topography has a unique solution if $Kh_{\max} \leq 1$, where $h_{\max}$ is the maximum depth of the layer, $K = \omega^2/g$, ω is the angular frequency of oscillation and g is the acceleration due to gravity. The proof uses the fact that when there are no bodies in the fluid, a line on which the potential is zero connects the free surface to infinity and, if $Ky \leq 1$ at every point on this line, uniqueness may be established.

Lack of uniqueness is equivalent to the existence of a 'trapped mode' (a local oscillation which has finite energy). McIver (1996) showed that trapped modes exist for pairs of surface-piercing bodies formed from parts of the streamlines associated with a pair of suitably place sources in deep water. Figures 1 and 2 illustrate the positions of bodies which support symmetric and antisymmetric trapped modes respectively. Although the uniqueness result of McIver (1998) is not applicable to bodies in deep water, it is instructive to note that in each case there are zero potential lines which asymptote to the lines $Ky = 1$ as $|x| \to \infty$, and which satisfy $Ky \leq 1$ everywhere. However this does not lead to a contradiction, because for the uniqueness proof to hold, vertical lines would need to be extended from all points on the free surface to the right of the bodies, to the zero potential line. This is impossible because the bodies are bulbous. Figures 1 and 2 also illustrate the position of the zero streamlines and in both cases they extend from infinity to a point on the free surface which is between the bodies. For symmetric motion the zero streamline is the line $x = 0$, whereas for antisymmetric motion, the lines asymptote to straight lines with gradient ± 1

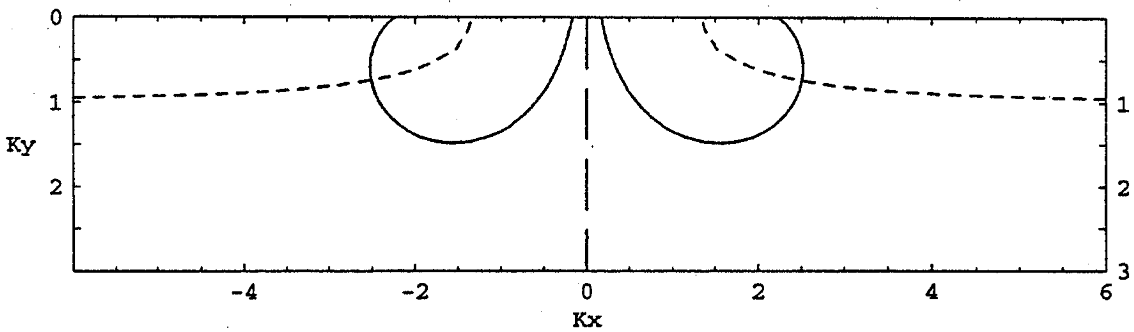

Figure 1 - Pair of surface-piercing bodies which support symmetric trapped modes;
- - - - zero potential line, ——— zero streamline

In this work a proof will be given that no symmetric trapped modes can be supported by a symmetric configuration of bodies which satisfies $n_x \leq 0$ at every point on the boundary of the bodies in the region $x \geq 0$, where n_x is the component of the inward normal to the structure in the x direction. Furthermore, it will be shown that the problem of whether antisymmetric trapped modes exist for this configuration reduces to the problem of whether trapped modes exist above a beach. In each case the position of the zero streamlines will be exploited.

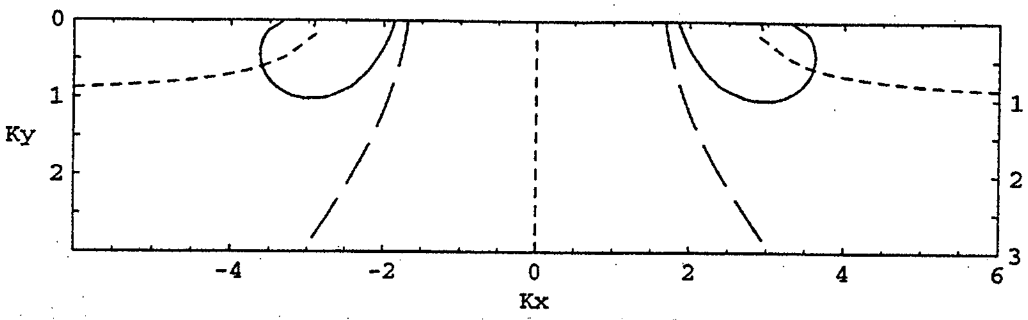

Figure 2 - Pair of surface-piercing bodies which support antisymmetric trapped modes; - - - - zero potential line, — — — zero streamline

Formulation

A symmetric system of bodies such as that illustrated in figure 3 is studied. The boundaries of the bodies are assumed to be piecewise smooth and to satisfy $n_x \leq 0$ at each point in the region $x \geq 0$, where n_x is the component of the inward normal in the x direction.

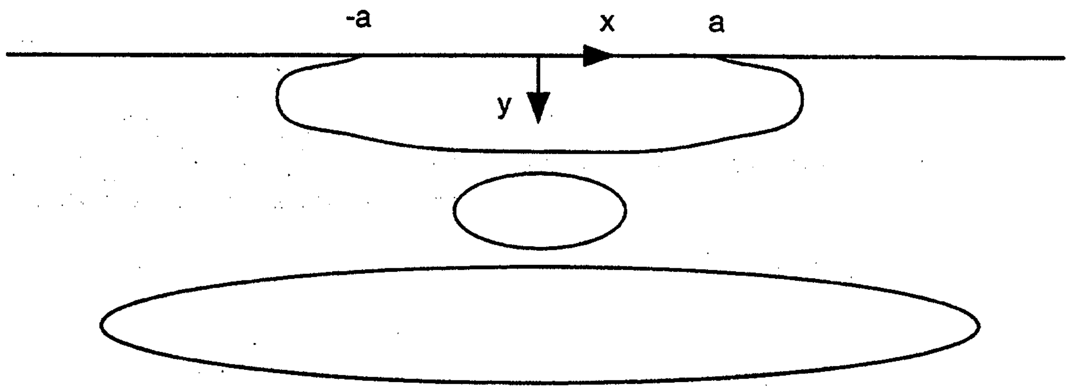

Figure 3 - Configuration of symmetric structures

The velocity potential which describes the two-dimensional, small oscillations of an inviscid and incompressible fluid at angular frequency ω is given by $Re[\phi(x,y)\,e^{-i\omega t}]$ where ϕ satisfies

$$\nabla^2\phi = 0, \text{ in the fluid} \tag{1}$$

and

$$K\phi + \frac{\partial\phi}{\partial y} = 0 \text{ on } y = 0. \tag{2}$$

Axes are chosen so that the origin is in the mean free surface (or inside the surface-piercing body) and the y-axis points vertically downwards. In addition, no flow through any rigid surface means that

$$\frac{\partial\phi}{\partial n} = 0 \text{ on the bodies.} \tag{3}$$

Trapped modes are defined to be non-zero solutions of the problem defined in (1)-(3) which have finite energy, ie which satisfy

$$\int_D |\nabla\phi|^2 \,\mathrm{dV} + K\int_F |\phi|^2 \,\mathrm{dx} < \infty, \tag{4}$$

where D is the fluid domain and F is the mean free surface. Uniqueness is established if the only solution to the homogeneous problem defined in (1)-(4) is the zero solution.

In the next section a proof that symmetric trapped modes do not exist for the configuration in figure 3 will be given.

Uniqueness of the symmetric motion

As the configuration of bodies is symmetric, the potential may conveniently be split into symmetric and antisymmetric parts, and the symmetric part of the potential satisfies $\partial\phi/\partial x = 0$ on that part of the line $x = 0$, which lies within the fluid. (In this section ϕ will be used to denote the symmetric potential.) Thus the line segments, together with the connecting pieces of the right-hand parts of the bodies, form a streamline. By suitable choice of the constant in the stream function ψ, this line may be taken to be a zero streamline, and it may be thought of as a rigid boundary to the fluid on its right.

Without loss of generality ϕ is assumed to be real. By definition, the function $\phi + i\psi$ is an analytic function of $x + iy$ and so ϕ and ψ satisfy the Cauchy-Riemann equations

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}, \quad \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}. \tag{5}$$

Furthermore

$$(\phi + i\psi)^2 = \phi^2 - \psi^2 + 2i\phi\psi \tag{6}$$

is analytic and so $\phi^2 - \psi^2$ and $2\phi\psi$ are harmonic and satisfy

$$\frac{\partial}{\partial x}(\phi^2 - \psi^2) = \frac{\partial}{\partial y}(2\phi\psi), \quad \frac{\partial}{\partial y}(\phi^2 - \psi^2) = -\frac{\partial}{\partial x}(2\phi\psi). \tag{7}$$

Green's theorem is applied to the harmonic functions $\phi^2 - \psi^2$ and x in the fluid region to the right of the zero streamline and yields

$$\int_{\partial D} (\phi^2 - \psi^2)n_x - x\frac{\partial}{\partial n}(\phi^2 - \psi^2)\,\mathrm{dS} = 0, \tag{8}$$

where ∂D consists of the zero streamline, the mean free surface in $x > a$ and a closing quarter circle at infinity. The quantity n_x is the component of the outward normal to the fluid region in the x direction and $\partial/\partial n$ denotes the outward normal derivative. It may be shown that ϕ and ψ decay at least as fast as $(x^2 + y^2)^{-1/2}$ as $x^2 + y^2 \to \infty$, $y \geq 0$ and so possible non-zero contributions to the integral in (8) may arise only from the zero streamline and the mean free surface. On the zero streamline $\psi = 0$ and $\partial\phi/\partial n = 0$ and so

$$\frac{\partial}{\partial n}(\phi^2 - \psi^2) = 2\phi\frac{\partial\phi}{\partial n} - 2\psi\frac{\partial\psi}{\partial n} = 0 \tag{9}$$

on this line, and

$$\int_{\psi=0} (\phi^2 - \psi^2)n_x - x\frac{\partial}{\partial n}(\phi^2 - \psi^2)\,\mathrm{dS} = \int_{\psi=0} \phi^2\, n_x\,\mathrm{dS}. \tag{10}$$

On the mean free surface $n_x = 0$, and from (7)

$$\frac{\partial}{\partial n}(\phi^2 - \psi^2) = -\frac{\partial}{\partial y}(\phi^2 - \psi^2) = \frac{\partial}{\partial x}(2\phi\psi). \quad \text{on } y = 0,\ x > a \tag{11}$$

Thus

$$\int_{y=0,\,x>a} (\phi^2-\psi^2)\,n_x - x\frac{\partial}{\partial n}(\phi^2-\psi^2)\mathrm{dx} = -\int_{y=0,\,x>a} 2x\frac{\partial}{\partial x}(\phi\psi)\,\mathrm{dx} = \int_{y=0,\,x>a} 2\phi\psi\,\mathrm{dx}, \tag{12}$$

where integration by parts and the fact that $\psi = 0$ on the zero streamline and $\psi \to 0$ as $x \to \infty$, $y = 0$ have been used. From the free surface boundary condition (2) and the Cauchy-Riemann equations (5)

$$2\phi\psi = -\frac{2}{K}\psi\frac{\partial\phi}{\partial y} = \frac{2}{K}\psi\frac{\partial\psi}{\partial x} = \frac{1}{K}\frac{\partial}{\partial x}(\psi^2). \tag{13}$$

Substitution of (13) in (12) and integration shows that the contribution to the integral in (8) from the mean free surface is zero. Thus from (8) and (10)

$$\int_{\psi=0} \phi^2\,n_x\,\mathrm{dS} = 0. \tag{14}$$

By assumption $n_x \le 0$ everywhere on the line $\psi = 0$ and so (14) may only be satisfied if ϕ is identically equal to zero on this line. This means that both ϕ and $\partial\phi/\partial n$ are zero on this line and so an application of Green's theorem to ϕ and the free surface Green's function yields that ϕ equals zero everywhere in the fluid region.

The antisymmetric motion

The antisymmetric part of the potential is dominated by the lowest antisymmetric wave-free potential as $x^2+y^2 \to \infty$. The corresponding stream function satisfies

$$\psi = p\left[-\frac{\cos 3\theta}{r^3} - \frac{K}{2}\frac{\cos 2\theta}{r^2}\right] + O\left(\frac{1}{r^4}\right) \tag{15}$$

as $r \to \infty$, for some constant p, where $x = r\sin\theta$ and $y = r\cos\theta$. A similar argument to that used by McIver (1998) may be used to show that zero streamlines asymptote to lines parallel to the lines $\theta = \pm\pi/4$, (ie lines with gradient ± 1), as $r \to \infty$, and that these lines must terminate on the mean free surface in a symmetric fashion. Thus a region contained between the free surface and a beach is formed and, if uniqueness may be established for that region, then by analytic continuation, the antisymmetric potential will be unique in the whole fluid domain.

Conclusion

In this work it has been shown that no symmetric trapped modes exist for certain symmetric configurations of bodies and that the question of whether antisymmetric trapped modes exist for these configurations may be reduced to the question of whether trapped modes exist above a beach of arbitrary shape. It is clear that the positions of the zero potential line and the zero streamlines are critical in determining whether trapped modes may exist, and future work will endeavour to exploit the nature of these lines and to determine more general conditions under which trapped modes exist.

References

McIver, M. 1996 'An example of non-uniqueness in the two-dimensional linear water wave problem.' *J. Fluid Mechanics*, Vol. 315, pp 257 - 266.

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