Line integrals on the free surface in ship-motion problems

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Very recently, one important result obtained in [1] relates the singular and highly-oscillatory properties of the ship-motion Green function. Further to the asymptotic analysis performed in this study which gives, in an analytically closed form, the complex-singular and highly-oscillatory terms of the ship-motion Green function when a field point approaches to the track of source point, we consider now the line integrals on the free surface involving these peculiar terms. It will be shown that the line integral at infinity disappears effectively and the waterline integral can be evaluated in an analytical way. This study is expected to be a signal of the beginning of a happy end to *the long march* during which a major stumbling block hindering the development of reliable and practical calculation methods is, as reported in [2] and [3], associated with these peculiarities of the ship-motion Green function and its subsequent integrations along the waterline on the free surface.

1 Properties of the ship-motion G.F. and line integrals on the free surface

The ship-motion Green function is expressed by the sum of a term corresponding to Rankine singularities and another accounting for free-surface effects. The free-surface-effect term can be further decomposed into a wave component and a local component which decreases rapidly in the far field, as shown in [4]. Furthermore, every wave system of the wave component associated with each dispersion curve is expressed in [5] in a simple and analytical form. According to the study presented in [6], we may classify three classes of unsteady ship waves. One associated with a closed dispersion curve is called ring waves. Another associated with the portion of open dispersion curves limited between two inflexion points located symmetrically in the upper and lower half Fourier plane, is usually called transverse waves. Both classes of ship waves behave in a similar way such as their amplitude decreases at a rate like $O(h^{-\frac{1}{2}})$ for large values of horizontal distance h from the source, although the transverse waves are limited within a wedge while the ring waves propagate out in all directions for limited values of the Brard number $\tau<1/4$. The third class of unsteady ship waves is associated with the portions of open dispersion curves from the inflexion points to infinity, and usually called divergent waves. The divergent waves behave in a particular way. Although the decreasing rate is the same as $O(h^{-\frac{1}{2}})$ in a given direction within the wedge and $O(h^{-\frac{1}{3}})$ along the wedge as well as the transverse waves, their amplitude decreases exponentially for a field point (ξ,η,ζ) approaching to the track of an *immerged* source point $(x,y,z<0)$. Furthermore, when a field point approaches to the track of a source located *at* the free surface ($z=0=\zeta$ and $|\eta-y|\to 0$), the divergent waves are highly oscillatory with infinitely increasing amplitude and infinitely decreasing wavelength.

These peculiar properties of the ship-motion Green function is analyzed in [1] by developing asymptotic expansions of the open dispersion curves at large wavenumber. The asymptotic analysis of the wave component contributed by the leading asymptotic term of a parabolic form shows that unsteady ship waves are highly oscillatory and singular when a field point approaches to the track of the source point at the free surface. The highly-oscillatory property and complex-singular behavior of unsteady ship waves are further expressed by (23), (26) and (27) in [1] in an original and analytically closed form. We rewrite the leading term given by (23a) in [1] here for $\zeta+z=0$

$$\widetilde{\mathcal{G}}=\frac{e^{-i\tau X}}{\sqrt{\pi}F^2}Y^{-\frac{1}{2}}\cos\left(\frac{X^2}{4Y}+\frac{Y}{2}+\frac{\pi}{4}\right) \tag{1}$$

in which $(X,Y)=(\xi-x,|\eta-y|)/F^2$, $F=U/\sqrt{gL}$ is the Froude number with U ship's speed, L ship's length and g the acceleration of gravity and $\tau=\omega U/g$ with ω the wave encounter frequency. Noting that the term $\widetilde{\mathcal{G}}$ given by the expression (1) exists only for $X<0$, i.e. $\xi<x$ in the downstream of the source point.

Within the framework of solving boundary-value problems governed by the Laplace equation, the velocity potential $\phi(\xi,\eta,\zeta)$ can be represented by the integrals over all boundary surfaces including the body boundary, the free surface and a fictitious surface enclosing the body and at infinity. For the ship-motion problem, the free-surface integral is further converted into two line integrals by using the Stokes' theorem

$$\phi_W+\phi_\infty=(\int_W-\int_\infty)\left[(F^2\phi_x+i2\tau\phi)G-F^2\phi G_x\right]t_y\,dl \tag{2}$$

where $\phi_W=\int_W(\cdot)\,dl$ stands for the waterline integral and $\phi_\infty=-\int_\infty(\cdot)\,dl$ the line integral at infinity. Furthermore, dl represents the differential element of arc length and t_y the component of the unit vector $\vec{t}=(t_x,t_y,0)$

tangent to the waterline and the line at infinity, is oriented clockwise. As already noted, the ship-motion Green function G contains several components plus the singular and highly-oscillatory term $\widetilde{\mathcal{G}}$ represented by (1). It is assumed that the terms other than $\widetilde{\mathcal{G}}$ do not induce any difficulty in both mathematical analyzes and numerical evaluations, so that only the term $\widetilde{\mathcal{G}}$ given by (1) are used in the following asymptotic analysis of the line integral at infinity and in the evaluation of influence coefficients corresponding to the waterline integral.

2 Asymptotic analysis of the line integral at infinity

In previous studies, the argument that $\phi \to 0$ at infinity as required by the radiation condition is usually used to say that the line integral ϕ_∞ in (2) vanishes. More elaborately, it is assumed that both ϕ and G decrease at the rate of order $O(h^{-a})$ and the integrals at infinity (surface or line integrals) disappear formally for $a=1$ in the case without free surface effects, and for any values of $a>0$ with free surface effects ($a=1/2$ for ring waves for example) via an analysis using the method of stationary phase. We have, however, a singular and highly-oscillatory term (1) included in G such that the methods used previously are not applicable, and that previous analyzes may not be complete.

To complete the task, we perform an asymptotic analysis of the line integral at infinity by considering a closed curve of rectangular form with length sides located in $-A \le x \le A$ at $y=\pm B$ and width sides located in $-B \le y \le B$ at $x=\pm A$. The line integrals along the length sides $y=\pm B$ are nil since $t_y=0$, hence

$$\phi_\infty(\xi,\eta) = \int_{-B}^{B} \left([(F^2\phi_x + i2\tau\phi)G - F^2\phi G_x]_{x=A} - [(F^2\phi_x + i2\tau\phi)G - F^2\phi G_x]_{x=-A} \right) dy \tag{3}$$

along the width sides $x=\pm A$ only. In (3), the values of (ξ,η) are those over ship's hull or a field around the ship, i.e. $\sqrt{\xi^2+\eta^2} \ll (A \text{ or } B)$ so that we may take $\xi=0=\eta$ without loss of generality. Along the upstream side $x=A$, ϕ decreases at the rate of $O(h^{-1})$ for the local component or $O(h^{-1/2})$ for ring waves at $\tau<1/4$ while $X=(\xi-x)/F^2=-A/F^2$ so that $\widetilde{\mathcal{G}}$ by (1) is applicable. Along the downstream side $x=-A$, the Green function G behaves like $O(h^{-1})$ for its local component or leading terms of ring waves of order $O(h^{-1/2})$ which exist for $\tau<1/4$, and G doesn't contain the term $\widetilde{\mathcal{G}}$ since $X=(\xi-x)/F^2=A/F^2>0$. As already noted, we are limited to analyze only line integrals involving the term $\widetilde{\mathcal{G}}$ so that the line integral at infinity (3) is estimated as

$$|\widetilde{\phi}_\infty| < C_0|I_0| + C_1|I_1| \quad \text{with} \quad I_0 = \int_{-B}^{B} \widetilde{\mathcal{G}}\, dy \quad \text{and} \quad I_1 = \int_{-B}^{B} \widetilde{\mathcal{G}}_x\, dy \tag{4}$$

along the upstream side $x=A$, where C_0 and C_1 dependent on values of F, τ and distributions of ϕ and ϕ_x along the side $x=A$ are assumed to be of order $O(A^{-1/2})$ since the leading terms in ϕ and ϕ_x are of order $O(h^{-1/2})$ with $h=\sqrt{A^2+y^2}$ along the upstream side $X=A$ as foregoing analyzed. Now the question is whether I_0 and I_1 are finite. Introducing (1) for $\widetilde{\mathcal{G}}$ into I_0 and I_1 of (4), we have

$$I_0 = \frac{e^{i\tau A/F^2}}{\sqrt{\pi}/2} \int_0^{B/F^2} \cos\left(\frac{(A/F^2)^2}{4Y} + \frac{Y}{2} + \frac{\pi}{4} \right) Y^{-\frac{1}{2}}\, dY \tag{5}$$

and

$$F^2 I_1 = i\tau I_0 + \frac{e^{i\tau A/F^2} A}{\sqrt{\pi} F^2} \int_0^{B/F^2} \sin\left(\frac{(A/F^2)^2}{4Y} + \frac{Y}{2} + \frac{\pi}{4} \right) Y^{-\frac{3}{2}}\, dY \tag{6}$$

Using the change of integral variable $Y=\tilde{A}e^u$ with $\tilde{A}=A/(F^2\sqrt{2})$, both integral $I_0(A)$ defined by (5) and $I_1(A)$ by (6) can be evaluated analytically for $B\to\infty$

$$I_0 = -2\sqrt{\pi} e^{i\tau A/F^2} \sqrt{\tilde{A}}\, \mathrm{J}_{1/2}(\tilde{A}) \quad \text{and} \quad F^2 I_1 = i\tau I_0 - \sqrt{2\pi} e^{i\tau A/F^2} \sqrt{\tilde{A}}\, \mathrm{Y}_{1/2}(\tilde{A}) \tag{7}$$

where $\mathrm{J}_{1/2}$ and $\mathrm{Y}_{1/2}$ are the Bessel functions defined in [7]. It follows from the asymptotic expansions (9.2.1) for $\mathrm{J}_{1/2}$ and (9.2.2) for $\mathrm{Y}_{1/2}$ in [7] that the absolute values of I_0 and I_1 are

$$|I_0| = 2\sqrt{2} \quad \text{and} \quad F^2|I_1| = 2\sqrt{1+2\tau^2} \tag{8}$$

effectively finite for $A\to\infty$ so that the line integral given in (4) $|\widetilde{\phi}_\infty| = O(A^{-1/2}) \to 0$ at infinity, since C_0 and C_1 in (4) are of $O(A^{-1/2})$ as already noted. Another integral on a fictitious surface at infinity can be analyzed in a similar way by using the expression (23a) given in [1] to express the highly-oscillatory term. This surface integral at infinity is expected to disappear as well since the singularity of the integrand is much weaker than the present line integral. This comfortable result is desirable and confirms that the velocity potential is correctly represented by source and dipole distributions on body boundary surface and along the waterline.

3 Analytical evaluation of the waterline integral

The waterline integral given in (2) is now considered. For the sake of simplicity, we denote $\sigma = F^2\phi + i2\tau\phi$ and $\delta = -\phi$ along the waterline and write the waterline integral in (2) by

$$\phi_W = \int_W \sigma G\, dy + \int_W \delta F^2 G_x\, dy \tag{9}$$

Furthermore, we suppose a linear distribution of σ and δ as

$$\sigma = \sigma_0 + \sigma_1 Y \quad \text{and} \quad \delta = \delta_0 + \delta_1 Y \tag{10}$$

along a straight segment described by $X = X_0 + X_1 Y$ within $Y_0 \le Y \le Y_1$ where $X < 0$ is assumed. Introducing (1) for $\widetilde{G}$ and (10) for σ and δ into (9), we have

$$\begin{aligned}\widetilde{\phi}_W = {} & \frac{e^{-i\tau X_0}}{\sqrt{\pi}} \int_{Y_0}^{Y_1} e^{-i\tau X_1 Y} \cos\left(\frac{(X_0+X_1Y)^2}{4Y} + \frac{Y}{2} + \frac{\pi}{4}\right) \left[(\sigma_0 + i\tau\delta_0)Y^{-\frac{1}{2}} + (\sigma_1 + i\tau\delta_1)Y^{\frac{1}{2}}\right] dY \\ & + \frac{e^{-i\tau X_0}}{2\sqrt{\pi}} \int_{Y_0}^{Y_1} e^{-i\tau X_1 Y} \sin\left(\frac{(X_0+X_1Y)^2}{4Y} + \frac{Y}{2} + \frac{\pi}{4}\right) \left[(\delta_0 X_0)Y^{-\frac{3}{2}} + (\delta_1 X_0 + \delta_0 X_1)Y^{-\frac{1}{2}} + (\delta_1 X_1)Y^{\frac{1}{2}}\right] dY\end{aligned} \tag{11}$$

whose numerical evaluations are not a easy task. Indeed, the highly-oscillatory behavior of the integrand in (11) induces large numerical errors due to dramatic cancellations between very large values with opposite signs in using a quadrature algorithm. The approximation in usual approaches to represent a segment of waterline by its centroids or any other points is simply wrong, according to the analysis in [3]. A rational and robust way to evaluate (11) is to perform integrations analytically. In fact, (11) can be expressed by

$$\begin{aligned}\sqrt{\pi} e^{i\tau X_0} \widetilde{\phi}_W = {} & \left(\frac{\bar{K}_{-\frac{1}{2}}(A_0, B_0 Y)|_{Y_0}^{Y_1}}{E^-} - \frac{K_{-\frac{1}{2}}(A_1, B_1 Y)|_{Y_0}^{Y_1}}{E^+}\right) \frac{\delta_0}{2i} \\ & + \frac{\bar{K}_{\frac{1}{2}}(A_0, B_0 Y)|_{Y_0}^{Y_1}}{E^-\sqrt{A_0 B_0}} \left(\frac{\sigma_0 + i\tau\delta_0}{2} + \frac{\delta_1 X_1}{4i}\right) + \frac{K_{\frac{1}{2}}(A_1, B_1 Y)|_{Y_0}^{Y_1}}{E^+\sqrt{A_1 B_1}} \left(\frac{\sigma_0 + i\tau\delta_0}{2} - \frac{\delta_1 X_1}{4i}\right) \\ & + \frac{\bar{K}_{\frac{3}{2}}(A_0, B_0 Y)|_{Y_0}^{Y_1}}{E^-\sqrt{A_0 B_0^3}} \left(\frac{\sigma_1 + i\tau\delta_1}{2} + \frac{\delta_0 X_1 + \delta_1 X_0}{4i}\right) + \frac{K_{\frac{3}{2}}(A_1, B_1 Y)|_{Y_0}^{Y_1}}{E^+\sqrt{A_1 B_1^3}} \left(\frac{\sigma_1 + i\tau\delta_1}{2} - \frac{\delta_0 X_1 + \delta_1 X_0}{4i}\right)\end{aligned} \tag{12}$$

in which $E^{\pm} = e^{\pm i(X_0 X_1/2 + \pi/4)}$ and the following notations are used

$$A_0 = |X_0|\sqrt{X_1^2/4 + 1/2 - \tau X_1}\,, \quad B_0 = (2/|X_0|)\sqrt{X_1^2/4 + 1/2 - \tau X_1}$$

$$A_1 = |X_0|\sqrt{X_1^2/4 + 1/2 + \tau X_1}\,, \quad B_1 = (2/|X_0|)\sqrt{X_1^2/4 + 1/2 + \tau X_1}$$

Furthermore,

$$K_\mu(A, BY)|_{Y_0}^{Y_1} = K_\mu(A, BY_1) - K_\mu(A, BY_0) \quad \text{for} \quad \mu = -1/2, 1/2, 3/2 \tag{13}$$

and $\bar{K}_\mu(u, v)$ is the complex conjugate of $K_\mu(u, v)$ which is defined by

$$K_\mu(u, v) = u^{\frac{1}{2}} \int_0^v t^{\mu-1} e^{-iu(t+1/t)}\, dt = (iu)^\mu u^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{u^{2n}}{n!} \Gamma(-\mu-n, iu/v) \tag{14}$$

where $\Gamma(\alpha, w)$ with $\alpha = -(\mu+n)$ and $w = iu/v$ is the complementary incomplete Gamma function defined by (6.5.3) in [7]. The Gamma function $\Gamma(\alpha, w)$ can be evaluated by using the series developments (6.5.29) in [7] for small to moderate values of $|w|$ and the asymptotic expansions (6.5.32) in [7] for large values of $|w|$.

4 Discussions and conclusions

In this study, we have first summarized the result obtained in [1] about the singular and highly-oscillatory properties of the ship-motion Green function. The leading term is rewritten by (1) when both field and source points are located at the free surface, and contained in the ship-motion Green function for the field point close to the track of the source point but not necessarily far away in the downstream. One interesting feature of (1) is that its dependence on the parameter τ is as simple as a modification by multiplying $\exp(-i\tau X)$ of the corresponding term for the Neumann-Kelvin steady flows which has been studied in [8] and [9], since other variables involved in (1) are independent of the frequency $f = \omega\sqrt{L/g}$ (and $\tau = fF$). This simple result is explained in [1] to be associated with the fact that the leading term of asymptotic expansions of open dispersion curves at all values of τ is of a parabola symmetrical with respect to the line $F^2\alpha = \tau$ in the Fourier plane, and that the translation of the origin of Fourier plane to $(\tau, 0)$ yields this oscillatory factor. The real and imaginary parts of $F^2\widetilde{\mathcal{G}}$ for $\tau = 1/5$ and at $X = -5$ are illustrated in Fig.1 by the solid and dashed lines, respectively.

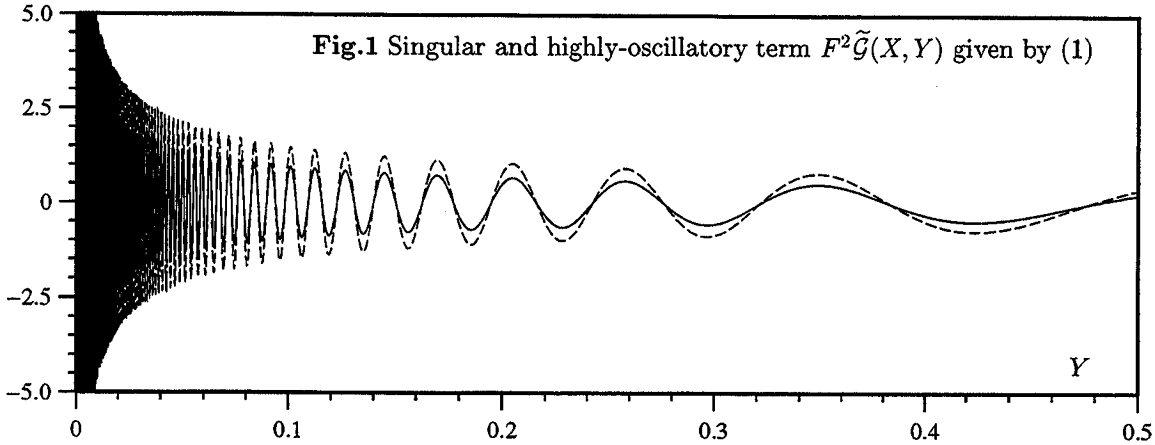

Fig.1 Singular and highly-oscillatory term $F^2\widetilde{\mathcal{G}}(X, Y)$ given by (1)

This new finding of the ship-motion Green function has motivated us to study the line integrals on the free surface arising from an application of the Stokes' theorem to the free-surface integral. The line integral far from the ship is analyzed by considering a rectangular contour. The asymptotic analysis of the integral along the contour confirm that the line integral at infinity involving $\widetilde{\mathcal{G}}$ disappears effectively. More critically, the line integral along the ship waterline is evaluated in an analytical way. The analytical integration of the singular and highly-oscillatory term involves a special function $K_\mu(u, v)$ which is *regular* and depicted in Fig.2a and Fig.2b for the real and imaginary parts, respectively. In both Fig.2(a,b), the solid, dashed and dot-dashed lines represent respectively $K_{-1/2}(u, v)$, $5K_{1/2}(u, v)$ and $50K_{3/2}(u, v)$ at the value $v = 1/5$.

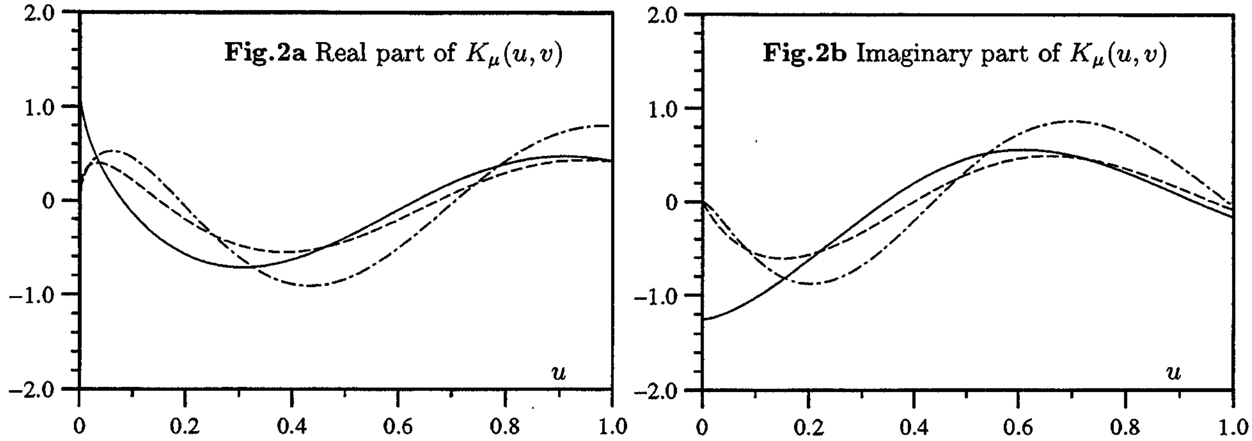

Fig.2a Real part of $K_\mu(u, v)$ **Fig.2b** Imaginary part of $K_\mu(u, v)$

The analytical expression (12) developed for a linear distribution of singularities along a straight segment is no singular and its extension to a more elaborate high-order representation of singularities and ship's hull and waterline geometry can be performed following the same spirit. Indeed, the present study provides critical and fundamental elements needed in implementing high-order (quadratic, for example) panel methods to solve the ship-motion problems.

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