Analogies for resonances in wave diffraction problems

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Introduction

It has been reported that on large numbers of equally spaced, bottom-mounted circular cylinders in line, large wave forces will be excited on each of the cylinders at particular wavenumbers close to those of trapped modes [1]. Similar observations have been made for circular arrays of bottom mounted circular cylinders, and these may be understood as "near-trapping" [2].

The existence of trapped modes for a bottom mounted circular cylinder placed on the centreline of a wave channel was first established by Callan et al. [3]. It is now known that they may arise not only at the wavenumbers below the cut-off value ($k^N d/\pi = 1/2$ for a Neumann boundary condition (B.C.) applied on the channel walls, and $k^D d/\pi = 1$ for a Dirichlet B.C., both for anti-symmetric waves with respect to the centreline of the channel, where k is the wavenumber and d is the half width of the channel); but also in the region above the cut-off wavenumber, or in the continuous spectrum [4, 5].

This paper first discusses other examples of trapped modes embedded in the continuous spectrum, e.g. when N bottom-mounted circular cylinders having the same radius a are equally spaced in line along the perpendicular plane to the channel walls in the wave channel. Evans and Porter [6] examined the trapped modes for multiple cylinders along the centreline of the channel, and observed the existence of up to N trapped modes below the cut-off wavenumber. In this paper, we will show numerically the existence of N trapped modes for the Neumann B.C., and up to N trapped modes for the Dirichlet B.C., for the case $a/s = 0.5$. In both cases, the trapped modes except that corresponding to the lowest frequency are shown to be embedded in the continuous spectrum region.

Next, an analogy is given between the trapped modes for a row of equally-spaced cylinders in the channel and the near-resonant modes for cylinders in the open sea. Another analogy with a spring-mass oscillating system is also given, which may offer some insights into such resonant phenomena in wave diffraction problems.

Trapped modes for equally spaced cylinders in a channel

The complex potential $\phi(x,y)$ which satisfies the Helmholtz equation $(\nabla^2 + k^2)\phi = 0$ is considered here. The boundary conditions on the channel walls are $\phi_y = 0$ on $y = \pm d$ (Neumann B.C.), or $\phi = 0$ on $y = \pm d$ (Dirichlet B.C.). A Neumann B.C. is applied on each cylinder, i.e., $\phi_{r_j} = 0$ on $r_j = a$, where the centre of each cylinder is placed at $(0, y_j)$, $y_j = -d + (2j-1)s$, $j = 1, \cdots, N$, and polar coordinates (r_j, θ_j) are employed with their origins at $(0, y_j)$. Moreover, the radiation condition $\phi \to 0$ for $x \to \pm\infty$ must be satisfied for trapped modes.

The multipole expansion method is employed here. The appropriate expressions for this problem can be found in McIver and Bennett [7] and in Linton and McIver [8]. The complex potential $\phi(x,y)$ is expressed in the following form:

$$\phi = \sum_{j=1}^{N} \sum_{n=0}^{\infty} Z_n (A_n^j \phi_n^j + B_n^j \psi_n^j), \tag{1}$$

where

$$\begin{aligned} \phi_n^j &= H_n(kr_j)\cos n\theta_j \\ &+ \frac{i^{n-1}}{2\pi}\int_{-\infty}^{\infty} \frac{e^{k\gamma(y-d)}(e^{-2k\gamma d} \pm e^{2k\gamma y_j}) + e^{-k\gamma(y-d)}(e^{-2k\gamma d} \pm e^{-2k\gamma y_j})}{\gamma \sinh 2k\gamma d} e^{-ikxt}\cosh n\tau dt, \end{aligned} \tag{2}$$

$$\begin{aligned} \psi_n^j &= H_n(kr_j)\sin n\theta_j \\ &+ \frac{i^{n}}{2\pi}\int_{-\infty}^{\infty} \frac{e^{k\gamma(y-d)}(e^{-2k\gamma d} \mp e^{2k\gamma y_j}) - e^{-k\gamma(y-d)}(e^{-2k\gamma d} \mp e^{-2k\gamma y_j})}{\gamma \sinh 2k\gamma d} e^{-ikxt}\sinh n\tau dt, \end{aligned} \tag{3}$$

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and $Z_n = J_n'(ka)/H_n'(ka)$, $\tau = \cosh^{-1} t$, $\gamma = \sinh \tau$. Also, the upper and lower signs of $\pm$ and $\mp$ correspond to Neumann and Dirichlet B.C. respectively, on the sides of the channel. Expanding the multipoles singular at one point $(0, y_j)$ about another point, $(0, y_p)$, and applying the boundary conditions on each of the cylinder surfaces, we obtain the following homogeneous systems of equations [8]:

$$A_m^p + \sum_{n=0}^{\infty}(\tilde{A}_n^p \alpha_{nm}^p + \tilde{B}_n^p a_{nm}^p) + \sum_{j=1,\neq p}^{N} \sum_{n=0}^{\infty}[\tilde{A}_n^j(C_{nm}^{jp} + \alpha_{nm}^{jp}) + \tilde{B}_n^j(E_{nm}^{jp} + a_{nm}^{jp})] = 0, \quad (4)$$

$$B_m^p + \sum_{n=0}^{\infty}(\tilde{A}_n^p \beta_{nm}^p + \tilde{B}_n^p b_{nm}^p) + \sum_{j=1,\neq p}^{N} \sum_{n=0}^{\infty}[\tilde{A}_n^j(D_{nm}^{jp} + \beta_{nm}^{jp}) + \tilde{B}_n^j(F_{nm}^{jp} + b_{nm}^{jp})] = 0, \quad (5)$$

where $p = 1, \cdots, N$; $m = 0, 1, \cdots$, in both cases, and the expressions $\tilde{A}_n^j = Z_n A_n^j$ and $\tilde{B}_n^j = Z_n B_n^j$ are used.

The infinite systems of equations are then truncated with $m, n = 0, 1, \cdots, M$. (In the following numerical computations, $M = 7$ has been employed). The symmetry of the trapped modes about the y-axis can be assumed [3], and thus $A_{2k+1}^j = B_{2k}^j = 0$ for $k = 0, 1, \cdots$ have been applied. Also, from the symmetry of the cylinder arrangements, it can easily be proved that $A_n^j = A_n^{N-j+1}$ and $B_n^j = -B_n^{N-j+1}$ for symmetric modes with respect to centreline of the channel; and that $A_n^j = -A_n^{N-j+1}$ and $B_n^j = B_n^{N-j+1}$ for anti-symmetric modes; so these have been employed to reduce the computations.

The complex determinants of the truncated systems of equations (4) and (5) have been calculated for various values of k, and it has been found that points exist where both real and imaginary parts of the complex determinants vanish. The homogeneous equations were then solved numerically in order to obtain non-trivial solutions for the values of k where the complex determinants vanish. Substituting the obtained values of k, A_n^j and B_n^j into Eq. (1), we have obtained numerically pure imaginary potentials, which indicate that we have obtained pure-trapped modes. The above procedure was repeated for the equations having only the imaginary parts of Eqs. (4) and (5), and we obtained the same results, which also shows that the complex potential becomes pure imaginary when the determinant vanishes.

The trapped wavenumbers at which the determinants vanish are shown in Table 1 for Neumann B.C. and in Table 2 for Dirichlet B.C., both for the case $a/s = 0.5$. In Tables 1 and 2, (s) indicates the symmetric mode of the corresponding trapped wave with respect to the centreline of the channel, and (a) indicates the anti-symmetric mode. Figure 1 shows equipotential contours of the trapped waves for $N = 4$. It can be seen that only the lowest trapped wavenumbers for each arrangement of cylinders are below the corresponding cut-off wavenumber in both Neumann and Dirichlet boundary conditions, and all except those are embedded in the continuous spectrum. The highest trapped mode in each case is equivalent to that for the case of one cylinder, since the same trapped wavenumber is obtained and the trapped wave satisfies $\phi_y = 0$ for a Neumann B.C. or $\phi = 0$ for a Dirichlet B.C. along the centreline between two adjacent cylinders. For a Neumann B.C., the second mode for four cylinders is also identical to the first mode for two cylinders. Similar relationships can be found between the trapped modes for five cylinders and those for ten cylinders. It should be noted that the trapped wavenumber $ks/\pi = 0.442869$ for a Dirichlet B.C. can not be obtained, in which case the non-existence of the trapped mode has already been proved [9].

Next, we focus our attention on the total wave forces induced on each cylinder. We have found that the coefficients B_1^j, which directly relate to the first-order force on cylinder j in the y-direction, precisely follow the formulae below:

$$B_1^{j(r)} = \sin \frac{r(2j-1)}{2N}\pi, \quad 1 \le j \le N; 1 \le r \le N; N \ge 1, \quad \text{(for Neumann B.C.)}, \quad (6)$$

$$B_1^{j(r)} = \cos \frac{(r+1)(2j-1)}{2N}\pi, \quad 1 \le j \le N; 1 \le r \le N-2; N \ge 3, \quad \text{(for Dirichlet B.C.)}, \quad (7)$$

where r is the mode number of the trapped mode.

Figure 2 indicates the distribution of wave forces at the trapped wavenumbers given by $ks/\pi = 0.442099$ ($r = 48$) and $ks/\pi = 0.439844$ ($r = 47$) for $N = 50$ with Dirichlet B.C. As seen in Figure 2, the distribution of forces in these trapped mode conditions for a row of cylinders in a wave "channel" with Dirichlet boundary conditions has close similarities with the forces on the arrays in the open seas discussed by Maniar and Newman [1]. Table 3 compares the trapped wavenumbers of the second highest modes in the channel with the wavenumber at which the peak load occurs within a finite array of N cylinders in the open sea. For a sufficient number of cylinders they agree very well, in spite of the different boundary conditions between these two cases.

Analogy with a spring-mass oscillating system

The basic unit in the mechanical analogue corresponds to one cylinder. It consists of a uniform massless cylindrical bar of unit cross-section, unit length and unit modulus elasticity, with a unit point mass attached at its mid-length. The mass is allowed to oscillate in the direction of the axis of the bar. If the two free ends of the bar are fixed, the square of the natural frequency of vibration of the point mass is 4, which may correspond to the trapped mode for one cylinder with Neumann B.C. The basic unit is now replicated N times in a straight line, by attaching the right hand end of one unit to the left hand end of the neighbouring unit. Both ends are then fixed (fixed B.C.) or allowed to be free to move (free B.C.). In both cases, we can deduce the eigenvalues of the discrete system (omitting the rigid-body mode of the free B.C. system) as [10]

$$\omega_r^2 = 2 - 2\cos(\frac{r\pi}{N}), \quad r = 1, \cdots, N(\text{for fixed B.C.}); \text{ and } r = 1, \cdots, N-1(\text{for free B.C.}). \tag{8}$$

The corresponding eigenvectors giving the displacements are

$$d_j^{(r)} = \sin\frac{r(2j-1)}{2N}\pi, \quad 1 \le j \le N; 1 \le r \le N; N \ge 1, \quad (\text{for fixed B.C.}), \tag{9}$$

$$d_j^{(r)} = \cos\frac{r(2j-1)}{2N}\pi, \quad 1 \le j \le N; 1 \le r \le N-1; N \ge 2, \quad (\text{for free B.C.}), \tag{10}$$

where index r designates the number of the mode, and the subscript j identifies the mass.

We see that such a mechanical model appears to display some of the characteristics of the array of cylinders in water waves. In particular, the distribution of displacements in the eigenmodes for the free B.C. case is identical to the distribution of forces on the cylinders at the Dirichlet mode trapped wavenumbers. These analogies open up several opportunities for the analysis of hydrodynamic resonances in periodic systems.

References

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Table 1 Trapped wavenumber ks/π for Neumann boundary conditions.

Mode number	1/6	2/7	3/8	4/9	5/10
1 cylinder	0.442869(a)	—	—	—	—
2 cylinders	0.248370(a)	0.442869(s)	—	—	—
3 cylinders	0.166245(a)	0.328444(s)	0.442869(a)	—	—
4 cylinders	0.124830(a)	0.248370(s)	0.366671(a)	0.442869(s)	—
5 cylinders	0.0999145(a)	0.199238(s)	0.296807(a)	0.388389(s)	0.442869(a)
10 cylinders	0.0499896(a)	0.0999145(s)	0.149699(a)	0.199238(s)	0.248370(a)
	0.296807(s)	0.343958(a)	0.388389(s)	0.425787(a)	0.442869(s)

Table 2 Trapped wavenumber ks/π for Dirichlet boundary conditions.

Mode number	1/6	2/7	3/8	4/9	5/10
1 cylinder	0.977759(a)	—	—	—	—
2 cylinders	0.977759(a)	—	—	—	—
3 cylinders	0.328444(a)	0.977759(a)	—	—	—
4 cylinders	0.248370(a)	0.366671(s)	0.977759(a)	—	—
5 cylinders	0.199238(a)	0.296807(s)	0.388389(a)	0.977759(a)	—
10 cylinders	0.0999145(a)	0.149699(s)	0.199238(a)	0.248370(s)	0.296807(a)
	0.343958(s)	0.388389(a)	0.425783(s)	0.977759(a)	—

Table 3 Comparison of the trapped wavenumber for Dirichlet B.C. and the near-resonant wavenumber for an array of N cylinders in the open sea (in ks/π, $a/s = 0.5$).

Number of cylinders, N	100	50	25	10
Trapped wavenumber	0.442676	0.442099	0.439844	0.425783
Near-resonant wavenumber[1]	0.442676	0.442104	0.439921	0.428557

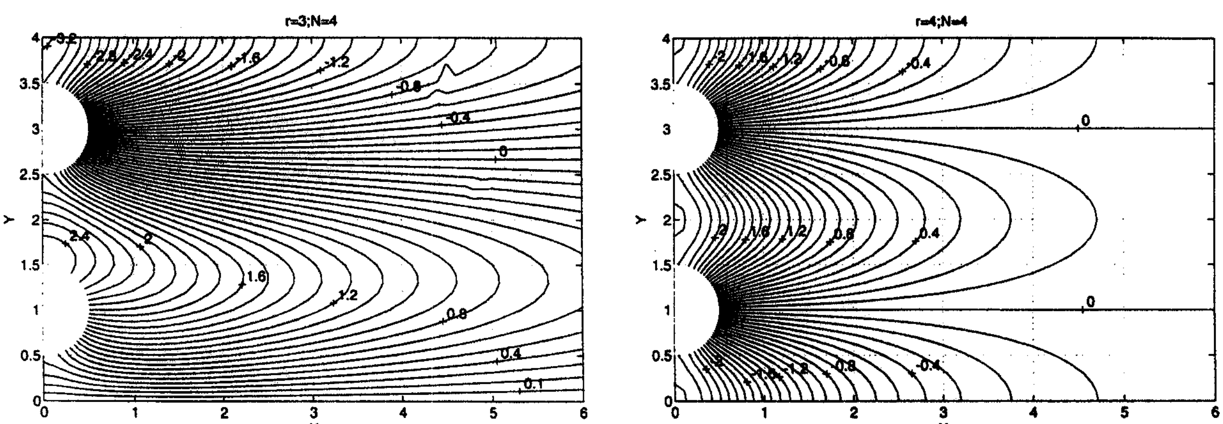

Figure 1 Equipotential contours of the trapped waves for $N = 4$.

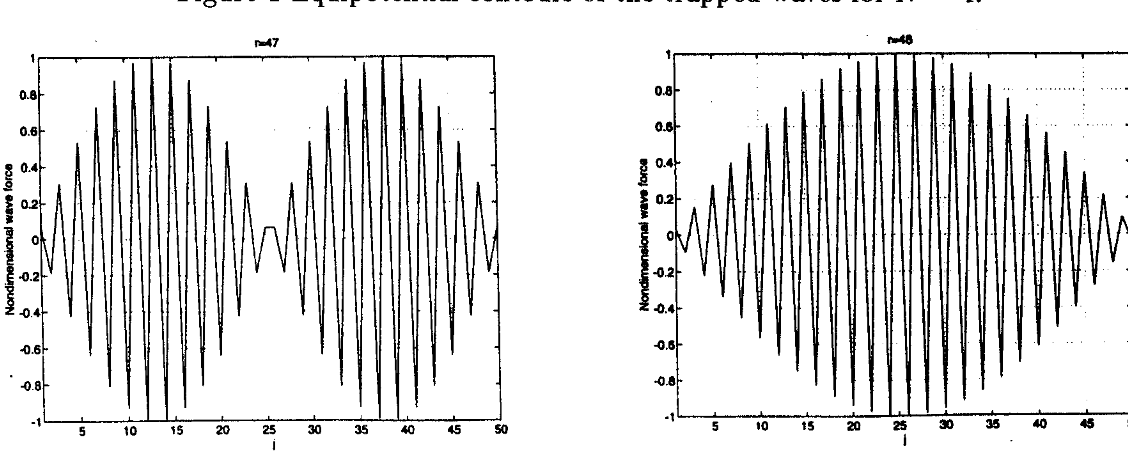

Figure 2 Distribution of wave forces for $N = 50$ with Dirichlet B.C.