FREE-SURFACE EVOLUTION AT THE EDGE OF AN IMPULSIVELY UPWELLING FLUID LAYER

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INTRODUCTION

Impulsive free-surface flows provide opportunities for studying analytically the early evolution of hydrodynamic nonlinearities. This type of research started with the wavemaker analysis of Peregrine (1972). The wavemaker problem is difficult because of the free-surface singularity arising at the intersection between the free surface and the moving plate. This singularity reveals that the asymptotic series is not uniformly valid. The Taylor series in time constitutes an outer expansion, and an inner expansion must be introduced to remove the singularity (King & Needham 1994). When the impulsive free-surface flow is due to submerged objects, one avoids free-surface singularities.

The present paper introduces a new variety of impulsive flow problems: a given flux through a fixed bottom is turned on at time zero. The present model is one of the cases where the exact surface elevation of a small-time expansion can be calculated to third order. Other examples are the submerged line vortex (Tyvand 1991), and the line source, either submerged (Tyvand 1992) or located at a bottom (Tyvand 1998). The solution for the submerged line source has been verified numerically by Kim (1997). A review of small-time expansions for impulsive free-surface flows is given by Tyvand & Miloh (1998).

We will investigate the evolution of nonlinear free-surface effects at the border between a region of forced uniform upwelling and a region of stagnant fluid. For small times the initial depth is the only characteristic length. This implies that all early nonlinear effects will be localized within a few length units around the edge between upwelling fluid and stagnant fluid. The early wave generation must also take place at the edge of the upwelling region. But the later wave propagation away from this edge cannot be captured by our analytical small-time expansion.

MATHEMATICAL FORMULATION

We consider an inviscid fluid layer of constant depth which is at rest at negative times. The layer depth h^* is the only length scale of the initial flow problem. So the Froude number F is defined as :

$$F = W^*(g^* h^*)^{-1/2} \tag{1}$$

The gravitational acceleration is g*, and W* is the upwelling fluid velocity plus the downwelling fluid velocity. In general, the fluid is upwelling along the bottom for positive x and downwelling for negative x. From now on we work with non-dimensional quantities based on the units W* and h*.

The inviscid flow is governed by Laplace's equation for the velocity potential $\Phi(x,y,t)$. The surface elevation is $\eta(x,t)$. The free-surface conditions are:

$$\partial\eta/\partial t + \nabla\Phi \cdot \nabla\eta = \partial\Phi/\partial y \qquad \text{at} \quad y = \eta(x,t) \tag{2}$$

$$\partial\Phi/\partial t + (1/2)|\nabla\Phi|^2 + F^{-2}\eta = 0 \qquad \text{at} \quad y = \eta(x,t) \tag{3}$$

As initial state we take an impulsive start from a situation at rest with a horizontal free surface:

$$\eta(x,0) = 0 \tag{4}$$

$$\Phi(x,0,0) = 0 \tag{5}$$

The dimensionless upwelling and downwelling velocities for positive and negative x will be denoted by V_+ and V_-, respectively. By definition we have $V_+ + V_- = 1$. The impulsively forced flow is given by:

$$\partial\Phi/\partial y = V_+ \quad , \; y=-1, \; x>0, \quad t>0 \tag{6a}$$

$$\partial\Phi/\partial y = -V_- \quad , \; y=-1, \; x<0, \quad t>0 \tag{6b}$$

RESULTS

The velocity potential and surface elevation are expanded as Taylor series in time (Tyvand 1991):

$$(\Phi,\eta) = H(t)\,[(\Phi_0,0) + t\,(\Phi_1,\eta_1) + t^2\,(\Phi_2,\eta_2) + ..] \tag{7}$$

H(t) denotes the Heaviside unit step function. We choose to develop the solution in terms of a Fourier series with an artificial periodicity of length L in x-direction. Then the first-order elevation is (sum taken over positive n):

$$\eta_1 = (V_+ - V_-)/2 + 2\pi^{-1} \sum_{n\ \mathrm{odd}} n^{-1}\,\mathrm{sech}(2\pi n/L)\,\sin(2\pi n x/L) \tag{8}$$

The exact solution in the limit L->∞ is:

$$\eta_1 = (V_+ - V_-)/2 + 2\pi^{-1} \sum_{k=1}^{\infty} (-1)^{k+1} \arctan[x/(2k-1)] \qquad (9)$$

This exact solution is found by differentiating the bottom source solution (Tyvand 1998). The convergence of eq.(8) is good when L>20. The second-order elevation consists of one odd and one even function of x:

$$\eta_2(x) = \eta_{2,odd} + \eta_{2,even} \qquad (10)$$

$$\eta_{2,odd} = \qquad (11a)$$

$$L^{-1}(V_- - V_+) \sum_{n\ odd} \operatorname{sech}(2\pi n/L) \tanh(2\pi n/L) \sin(2\pi n x/L)$$

$$\eta_{2,even} = \qquad (11b)$$

$$-(\pi L)^{-1} \sum_{n,m\ odd} \sum [\ (m^{-1}-n^{-1}) \tanh(2\pi(n-m)/L) \cos(2\pi(n-m)x/L)$$

$$- (m^{-1}+n^{-1}) \tanh(2\pi(n+m)/L) \cos(2\pi(n+m)x/L)\ ]$$

$$\operatorname{sech}(2\pi n/L) \operatorname{sech}(2\pi m/L)$$

Both these terms are important except for the case of antisymmetric upwelling/downwelling ($V_+ = V_- = 1/2$), where the odd term vanishes.

In this note we omit the third-order terms due to nonlinear interaction. We consider only the gravity-dependent term $\eta_3^{(F)}$, where superscript F refers to Froude number. It is proportional to the odd contribution to the second-order elevation:

$$3 F^2 \eta_3^{(F)} = 2\ \eta_{2,odd} \text{ (evaluated for pure upwelling)} \qquad (12)$$

In figure 1 some snapshots of the total surface elevation to third order is shown, for pure upwelling: $(V_+, V_-) = (1,0)$. The Froude number is 0.5.

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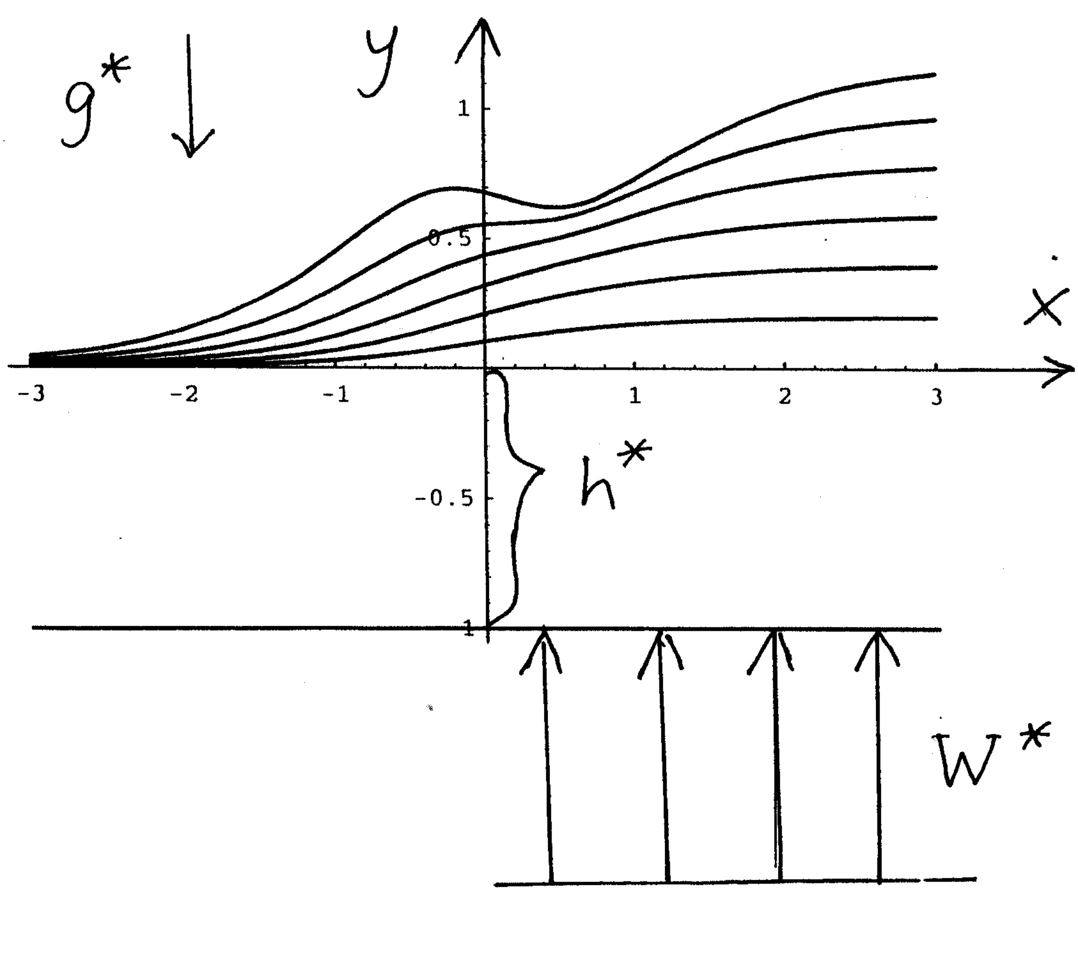

Figure 1: Snapshots of free surface shape $y=\eta(x,t)$, to third order in the small-time expansion. Pure upwelling: $(V_+, V_-)=(1,0)$. F=0.5. Increments of 0.2 are chosen in the dimensionless time t (= t W*/h*).*