On the Non-Uniqueness in the 2D Neumann–Kelvin Problem for a Tandem of Surface-Piercing Bodies

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1. Introduction

The present note is concerned with a tandem of horizontal cylindrical bodies moving forward with constant velocity U in the free surface of an inviscid, incompressible fluid under gravity. The resulting fluid motion is described by the linearized water-wave theory (the corresponding boundary value problem is usually referred to as the Neumann–Kelvin problem). For a totally submerged body in the fluid of infinite depth Kochin (1937) and Vainberg & Maz'ya (1973) had given almost exhaustive mathematical theory of this problem. The case of surface-piercing bodies is much more complicated. A number of significant results have been obtained for this case by Ursell (1981), Lenoir (1982), Kuznetsov & Maz'ya (1989), Kuznetsov & Motygin (1995), but a lot of questions still remains unsolved for it.

Treating the special case of semi-submerged circular cylinder Ursell (1981) found that the Neumann–Kelvin problem has a two-parameter set of solutions. He proposed two conditions complementing the original problem to make it well-posed (uniquely solvable for all values of U with possible exception for a sequence tending to zero). The corresponding "least singular" solution gives a bounded velocity field near corner points. This result was generalized by Kuznetsov & Maz'ya (1989), who proved that the least singular statement is well-posed for an arbitrary contour having non-acute angles with the free surface. A number of other supplementary conditions appeared in Lenoir (1982), Kuznetsov & Maz'ya (1989), Motygin & Kuznetsov (1995) and Kuznetsov & Motygin (1995). The "resistanceless" supplementary conditions considered in the last paper provide that the total resistance (a sum of wave resistance and spray resistance) vanishes for a surface-piercing tandem.

Recently McIver (1996) demonstrated the existence of a non-uniqueness example for the 2D sea-keeping problem. She applied the so-called inverse procedure for simultaneous construction of two surface-piercing bodies and of the potential of mode trapped by these bodies. Here we use the same method for the Neumann–Kelvin problem. Actually, our example delivers non-uniqueness to two statements of the problem, namely, to the least singular and to the resistanceless statements for a surface-piercing tandem.

2. Statement of the problem

The geometrical notations are given in figure 1.

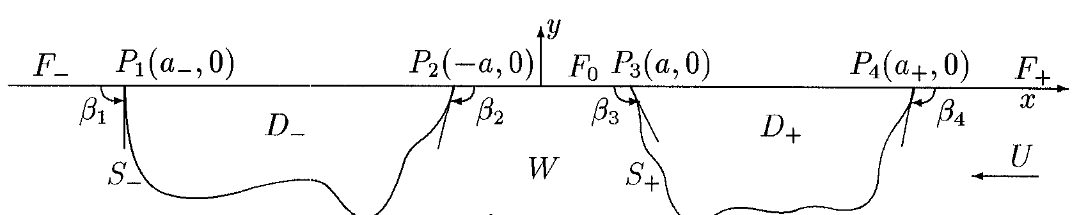

Figure 1

Assuming that the fluid motion is steady-state in a coordinate system attached to the tandem we describe it by a velocity potential u, which must satisfy the boundary value problem:

$$\nabla^2 u = 0 \text{ in } W, \quad u_{xx} + \nu u_y = 0 \text{ on } F_0 \cup F_+ \cup F_-, \quad \partial u/\partial n = U\cos(n,x) \text{ on } S_+ \cup S_-, \tag{1}$$

$$\lim_{x\to+\infty} |\nabla u| = 0, \qquad \sup\{|\nabla u| : (x,y) \in W \setminus E\} < \infty, \quad \int_{W\cap E} |\nabla u|^2 \,dx\,dy < \infty. \tag{2}$$

Here $\nu = g\,U^{-2}$, where g is the acceleration due to gravity; $S_\pm$ denotes an open arc lying in $\mathbb{R}^2_-$ and E is a compact set in $\overline{\mathbb{R}^2_-}$, containing $D_- \cup D_+$ with contiguous parts of F_0, F_+ and F_-.

The last condition in (2) allows to avoid strong singularities at the corner points P_k, $k = 1,2,3,4$, because according to Kuznetsov & Maz'ya (1989)

$$u = \begin{cases} C + B\rho^{\pi/2\beta}\sin(\pi\theta/2\beta) + A\rho\cos(\theta-\alpha) + O(\rho^{1+\delta}) & \text{when } \beta > \pi/2, \\ C + B[\rho\log\rho\sin\theta - \rho(\theta-\pi/2)\cos\theta] + A\rho\cos(\theta-\alpha) + O(\rho^{1+\delta}) & \text{when } \beta = \pi/2, \\ C + A\rho\cos(\theta-\alpha) + O(\rho^{1+\delta}) & \text{when } \beta < \pi/2. \end{cases} \quad (3)$$

as $\rho \to 0$. Here (ρ_k, θ_k) are polar coordinates with a pole at P_k and $(-1)^k \cdot \mathbf{i}$ directed along the polar axis. The angles $\theta_{1,3}$ $(\theta_{2,4})$ are measured counterclockwise (clockwise) and $0 \le \theta_k \le \beta_k$. The subscript k indicating the dependence of variables, coefficients and $\delta > 0$ on P_k is omitted.

If $\beta_k \ge \pi/2$ and $B_k \ne 0$, then the velocity vector ∇u is singular when approaching P_k along all non-horizontal directions. However, u_x has finite limits along the x-axis which will be denoted by $u_x(P_k)$. Following Ursell (1981) we say that u satisfying (1), (2) is the *least singular* solution (solution to Problem (L)) if every $B_k = 0$ in the asymptotics (3) for u.

Let us turn to the resistanceless statement of the Neumann–Kelvin problem for a tandem. We remind that any solution to (1), (2) has the following asymptotics as $|z| \to \infty$:

$$u(x,y) = C + Q\log(\nu|z|) + H(-x)\mathrm{e}^{\nu y}(\mathcal{A}\sin\nu x + \mathcal{B}\cos\nu x) + \psi(x,y). \quad (4)$$

Here $z = x + iy$, C is an arbitrary constant, H is the Heaviside function, and the estimates $\psi = O(|z|^{-1})$, $|\nabla\psi| = O(|z|^{-2})$ hold. The constants Q and $\mathcal{A}$ are determined by

$$\pi\nu Q + \sum_\pm [u_x(P_{3\pm1}) - u_x(P_{2\pm1})] = \nu\int_S \frac{\partial u}{\partial n}\,\mathrm{d}s,$$

$$-\frac{\mathcal{A}}{2} = \int_S \left[u\frac{\partial}{\partial n}(\mathrm{e}^{\nu y}\cos\nu x) - \frac{\partial u}{\partial n}\mathrm{e}^{\nu y}\cos\nu x\right]\mathrm{d}s + \sum_\pm \pm[\nu^{-1}u_x(x,0)\cos\nu x + u(x,0)\sin\nu x]_{x=\pm a}^{x=a_\pm},$$

where $\sum_\pm$ means summation of two terms. The last formula with cos and sin replaced by $-$sin and cos respectively gives the coefficient $\mathcal{B}$.

Let u satisfy (1), (2), and let the following supplementary conditions $\mathcal{A} = 0$, $\mathcal{B} = 0$, $u_x(P_1) = u_x(P_2)$, $u_x(P_3) = u_x(P_4)$ hold. Then we say that u is the *resistanceless* potential (solution to Problem (R)). The term *resistanceless* becomes clear if we take into account the formula expressing the total resistance to forward motion (see Motygin & Kuznetsov (1995)):

$$R = -\frac{\rho\nu}{4}(\mathcal{A}^2 + \mathcal{B}^2) - \frac{\rho}{2\nu}\left\{[u_x^2(x,0)]_{x=a_-}^{x=-a} + [u_x^2(x,0)]_{x=+a}^{x=a_+}\right\},$$

where ρ is fluid's density.

Using the source method proposed by Kuznetsov & Maz'ya (1989) one can prove that *problems (L) and (R) (with an arbitrary right hand side in the Neumann condition) are solvable for all $\nu > 0$ with possible exception for a discrete sequence of values (own for each problem).*

3. Non-uniqueness examples

For construction of examples we use the inverse procedure, which replaces finding a solution to a given problem by determining physically reasonable fluid region for a given solution. We define the latter with the help of Green's function

$$G(z,\zeta) = -\frac{1}{2\pi}\log(\nu^2|(z-\zeta)(z-\overline{\zeta})|) - \frac{1}{\pi}\int_0^\infty \frac{\cos k(x-\xi)}{k-\nu}\,\mathrm{e}^{k(y+\eta)}\,\mathrm{d}k - \mathrm{e}^{\nu(y+\eta)}\sin\nu(x-\xi),$$

which describes the forward motion of a source placed at $\zeta = \xi + i\eta$. Putting

$$u(z) = (\pi/\nu)\,[G_x(z,\pi/\nu) - G_x(z,-\pi/\nu)], \quad (5)$$

we obtain a solution to the problem (1), (2) for a surface-piercing tandem if at least one of the streamlines of the flow connects the x-axis on either side of a dipole point and another streamline similarly surrounds the other dipole point (we interpret these streamlines as contours of two bodies). The streamlines are level lines of the stream function v, which is a harmonic conjugate to u. We use the following representations:

$$\begin{aligned} v(z) &= \int_0^\infty \frac{\cos k(x+\pi/\nu) - \cos k(x-\pi/\nu)}{k-\nu}\, \mathrm{e}^{ky}\, \mathrm{d}k \\ &= \mathrm{Re}\left\{\mathrm{e}^{-i\nu z}\left[\mathrm{Ei}(i\nu(z-\pi/\nu)) - \mathrm{Ei}(i\nu(z+\pi/\nu))\right]\right\}, \end{aligned} \tag{6}$$

where the second formula in terms of the exponential integral follows from 8.212.5, Ryzhik & Gradshteyn (1980) *Table of Integrals.* The asymptotics of Ei implies that $v(z) \sim \pm \log|z \pm \pi/\nu|$ as $z \to \mp\pi/\nu$. Thus, the streamlines enclosing the dipoles do exist for sufficiently large values of v, and these lines are close to semicircles which are the level lines of $\log|z \pm \pi/\nu|$.

The particular combination of dipoles (5) is chosen to cancel wave terms in the asymptotics of u. The latter fact is an immediate consequence of the following formula

$$G(z;\zeta) \sim -\pi^{-1}\log(\nu|z|) - 2\mathrm{e}^{\nu(y+\eta)}\sin\nu(x-\xi) \text{ as } x \to -\infty.$$

Therefore, u delivers a solution to Problem (R), because the second pair of supplementary conditions in the definition of this problem is also fulfilled. Really, the direct calculation based on (6) shows that $v_y - \nu v = 0$ when $y = 0$ and $x \neq \pm\pi/\nu$, and hence, the derivatives $v_y = u_x$ have the same value at both end-points (belonging to the x-axis) of any streamline enclosing one of the dipoles.

Actually, u has no singular points on the x-axis except for $\pm\pi/\nu$. Hence, if $\beta_k \geq \pi/2$, $k = 1,2,3,4$ (see fig. 1 for definition of β_k) for the streamlines which we interpret as bodies, then u delivers a solution to Problem (L) as well. The latter is the case because we have

$$\tan\beta_k = (-1)^k u_y(P_k)/u_x(P_k) = (-1)^{k+1} v_x(P_k)/v_y(P_k).$$

From this and from the behavoiur of derivatives shown on figure 3(a) one obtains that all angles β_k are non-acute for the streamlines given by (6).

4. Discussion

A new type of non-uniqueness for the Neumann–Kelvin problem is described. The well-known non-uniqness (see Introduction) is a consequence of sub-definiteness of this boundary value problem for surface-piercing bodies, and occurs for all such bodies and all values of ν. The

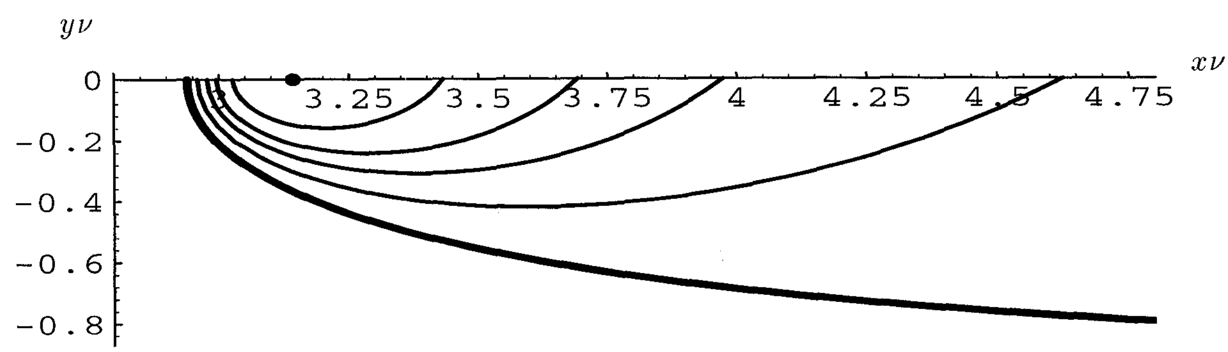

Figure 2. Streamlines for $v = 0$ (bold line), 0.2, 0.4, 0.6, 1.0.

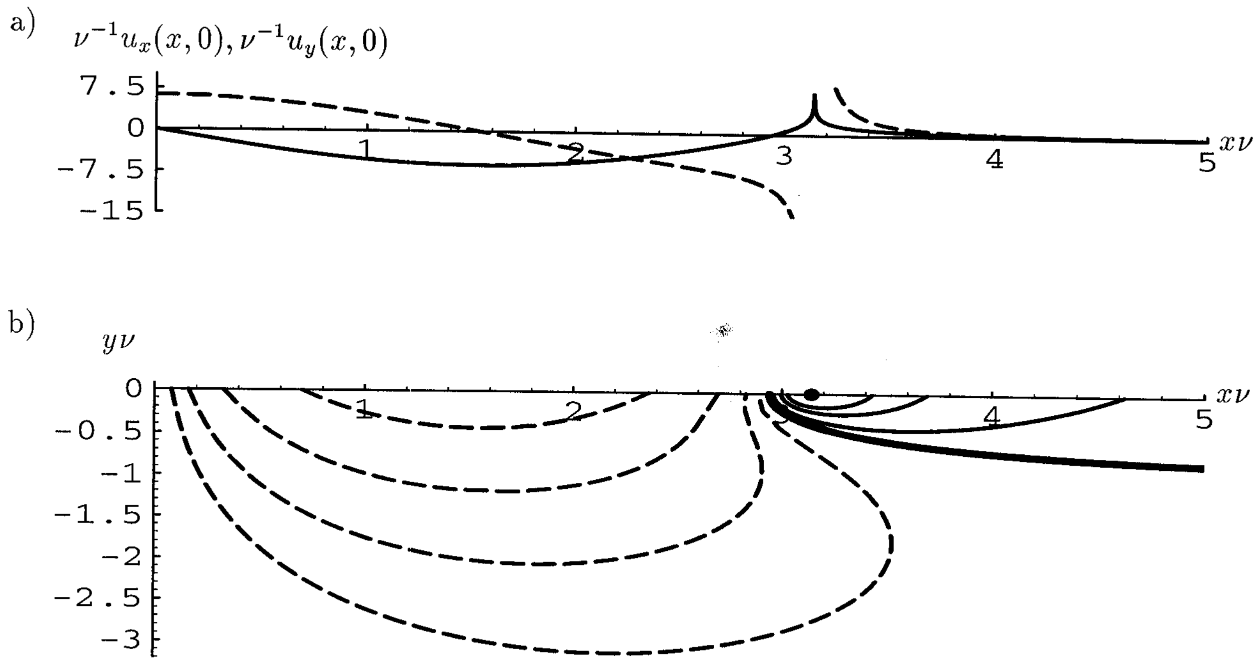

Figure 3. a) $\nu^{-1}u_x(x,0)$ and $\nu^{-1}u_y(x,0)$ (dashed line) are plotted against $x\nu$; b) shows streamlines for $v = -0.5, -1.0, -2.0, -4.0$ (dashed lines), $v = 0$ (bold line) and $v = 0.2, 0.6, 1.0$.

new type of non-uniqueness takes place only for special values of ν depending on the geometry. These values are point eigenvalues corresponding to modes of finite energy (known as trapped modes) embedded in the continuous spectrum of the relevant pseudo-differential operator. The latter spectrum is known to be $(0, +\infty)$.

We use a pair of horizontal dipoles for obtaining trapped modes, whereas McIver (1996) applies a pair of sources in her construction. The reason is that dipoles deliver an example for two statements simultaneously. The potential generated by two sources gives an example of non-uniqueness only for the least singular statement and cannot satisfy the second pair of supplementary conditions in Problem (R).

There is no unique set of supplementary conditions vanishing the total resistance to the forward motion of more than two surface-piercing bodies. At the same time, the least singular solution can be naturally defined for any number of bodies. The corresponding non-uniqueness examples can be easily constructed. In particular, the potential (5) delivers examples of non-uniqueness for Problem (L) with 3 and 4 cylinders (see fig. 3(b) for the corresponding streamlines).

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DISCUSSION

Tuck E.O.: Could you explain why you are studying the Neumann-Kelvin problem? This is a serious question, since there are matters raised in the paper such as singularities at the body-FS junction points, which relate directly to the question of the practical relevance of the N-K problem. Although my own opposition to the N-K problem is well known, it is possible that critics like me could be converted to believe in it, if studies like this were motivated to explain these singularities, or to use them as an outer expansion is a systematic approximation.

Motygin O., Kuznetsov N.: We consider the N-K problem as a phenomenological model. The so-called full non linear problem is also only a model, because it involves the assumption that the fluid motion is irrotational everywhere. However, this is hardly true near body-FS junction points. Since the linear N-K problem requires supplementary conditions, their choice can be used for an appropriate phenomenological description of fluid motion near junction points. Different supplementary conditions could be good for different ranges of the Froude number.