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Modelling of fully nonlinear internal waves and their generation in transcritical flow at a geometry

by

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Introduction

Knowledge of flows due to internal waves, their origin and propagation, is important for many reasons. Relevant examples are flows in fjords and at sills, breaking of internal waves and mixing processes in the ocean, motion in coastal water and sub-surface waves in a layered ocean. An important aspect of the latter relates to oil exploration in deep water, with operation performed from ships or oil platforms floating at the sea surface, connected to subsea drilling or production via long cables. Knowledge of currents in the ocean, which may be induced by internal waves, may be of importance for the design of such concepts, in addition to the wave effects at the ocean surface. Dynamics of internal waves is also important in dimensioning of subwater bridges, which have been proposed across Norwegian fjords. This study is in particular motivated by needs relating to the two latter problems. In this abstract we describe recent efforts at the University of Oslo on this issue, both theoretical and experimental.

Time stepping of the interface

In the two-layer model we study fully nonlinear two-dimensional motion of two fluid layers of infinite horizontal extension under the action of gravity, with the gravitation force along the negative vertical direction. The lower fluid layer has thickness h_1 at rest and constant density ρ_1, and the upper layer has thickness h_2 at rest and constant density ρ_2, where ρ_2 is smaller than ρ_1. Hereafter, index 1 refers to the lower fluid, and index 2 to the upper. A coordinate system $O - xy$ is introduced with the x-axis along the interface at rest and the y-axis pointing upwards. Unit vectors $\mathbf{i}, \mathbf{j}$ are introduced accordingly. We assume that the two fluids are homogenous and incompressible and that the motion in each of the layers is irrotational such that the velocities may be obtained by potential theory, i.e.

$$\mathbf{v}_1 = u_1\mathbf{i} + v_1\mathbf{j} = \nabla\phi_1, \qquad \mathbf{v}_2 = u_2\mathbf{i} + v_2\mathbf{j} = \nabla\phi_2, \tag{1}$$

where ϕ_1 and ϕ_2 satisfy the Laplace equation in their respective domains.

We adopt a pseudo Lagrangian method where pseudo particles are introduced on the interface, each with a weighted velocity given by

$$\mathbf{v}_\times = (1-\alpha)\mathbf{v}_1 + \alpha\mathbf{v}_2 \tag{2}$$

where $0 \leq \alpha \leq 1$. To determine the position $\mathbf{R} = (X, Y)$ of a pseudo particle we use

$$\frac{D_\times \mathbf{R}}{dt} = \mathbf{v}_\times \tag{3}$$

where a pseudo Lagrangian derivative is introduced by $D_\times/dt = \partial/\partial t + \mathbf{v}_\times \cdot \nabla$. From the dynamic boundary condition at the interface I we find

$$\frac{D_\times(\phi_1 - \mu\phi_2)}{dt} = \mathbf{v}_\times \cdot (\mathbf{v}_1 - \mu\mathbf{v}_2) - \frac{1}{2}(\mathbf{v}_1^2 - \mu\mathbf{v}_2^2) - (1-\mu)gY - \frac{\sigma}{\rho_1 R_I} \quad \text{at } I \tag{4}$$

where $\mu = \rho_2/\rho_1$. The equations (3) and (4) contain sufficient information to integrate $\mathbf{R}$ and $\phi_1 - \mu\phi_2$ forward in time. It is, however, an advantage to apply also higher order derivatives of (3) and (4) in a time stepping procedure for $\mathbf{R}$ and $\phi_1 - \mu\phi_2$.

The Eulerian velocity fields in the layers are obtained by solving the Laplace equation at each time step. It turns out that accurate solution of the Laplace equation is crucial to an algoritm for computing interfacial flows. Earlier works on time evolution of nonlinear interfacial waves have applied singularity distributions directly along the interface to solve the Laplace equation. We have sought a different method, and have chosen to employ Cauchy's integral theorem for this purpose, which is advantageous in avoiding instability.

Invoking complex analysis we introduce complex variable $z = x + iy$ and complex velocities $q_j(z) = u_j - iv_j$, $j = 1, 2$. Since q_j are analytic functions of z we have by use of Cauchy's integral theorem

$$-\pi i q_2(z') = PV \int_I \frac{q_2(z)dz}{z'-z} + \int_I \frac{q_2(z)^* dz^*}{z^* + 2ih_2 - z'} \qquad (z \text{ on } I) \tag{5}$$

$$\begin{aligned} \pi i q_1(z') &= PV \int_I \frac{q_1(z)dz}{z'-z} + \int_I \frac{q_1(z)^* dz^*}{z^* - 2ih_1 - z'} \\ &+ \int_B \frac{q_1(z)dz}{z'-z} + \int_B \frac{q_1(z)^* dz^*}{z^* - 2ih_1 - z'} \qquad (z \text{ on } I) \end{aligned} \tag{6}$$

$$\begin{aligned} \pi i q_1(z') &= \int_I \frac{q_1(z)dz}{z'-z} + \int_I \frac{q_1(z)^* dz^*}{z^* - 2ih_1 - z'} \\ &+ PV \int_B \frac{q_1(z)dz}{z'-z} + \int_B \frac{q_1(z)^* dz^*}{z^* - 2ih_1 - z'} \qquad (z \text{ on } B) \end{aligned} \tag{7}$$

where PV denotes principal value and B denotes the boundary of a geometry in the lower fluid. Only the real part of the principal value integrals in (5)–(7) are singular.

Transcritical flow at a topography. Upstream solitary waves

We apply the model to study transcritical two-layer flow at a bottom topography. There are several questions concerning this subject: Under which conditions is the flow unsteady? Another aspect is upstream influence in stratified flows, which in part can be addressed by the present two-layer model. Furthermore, for which conditions may transcritical flow over topography generate upstream solitary waves? These topics have been discussed in earlier works exploiting hydraulic nonlinear theory or weakly nonlinear dispersive models. These methods have, however, limited validity with regard to nonlinearity and dispersion and give unrealistic predictions for finite amplitude and moderate wave length.

In the transcritical regime we find that an undular upstream bore is generated when the speed of the geometry, U, is less than a value which slightly exceeds the linear long wave speed, c_0. In the remaining part of the transcritical regime we find that solitary waves propagating upstream are generated by the geometry. We show an example in figure 1, which is due to a half elliptical bottom topography with horizontal half-axis $10h_1$, vertical half-axis $0.1h_1$, moving with speed $U/c_0 = 1.1$ in the lower layer, with $h_2/h_1 = 4$ and $\mu = 0.7873$. We have performed a very long time simulation with this configuration. A depression behind the moving geometry stabilizes at a level of 80% of the initial thickness of the lower fluid. The upstream waves have all the same amplitude, within a variation of 0.3%. The amplitude has same magnitude as the depth of the thinner layer, which means that the nonlinear effect is rather strong. Upon comparing with the solution of a steady profile we find a very good agreement between the computed profiles and wave speeds. Thus, the simulated waves may be regarded as a train of solitary waves.

In several other examples (not shown) we find that a moving geometry generates upstream disturbances with rather large elevation, even for geometries with small height (the volume of the geometry cannot be too small). We also compare our results with weakly nonlinear Korteweg-de Vries, finite depth and Benjamin-Ono theories. Our results indicate that these theories in many cases predict quite unrealistic wave profiles, and that a fully nonlinear method in general is required to investigate stratified transcritical flow at a geometry or bottom topography.

Experiments

We also perform experiments on internal waves with the perposes to determine wave shapes, velocity profiles and compare with theoretical models, such as the interface method. The experiments are carried out in a wave tank, and we use fresh water above salt water with vertical density profiles varying between $\rho_2 = 1.0000 g/cm^3$ and $\rho_1 = 1.0225 g/cm^3$. The velocity field in solitary waves is measured using Particle Tracking Velocimetry, where the fluid is seeded with particles and the motion is recorded onto a video tape. This is later digitized and analyzed by image processing.

The experiments are carried out with different (vertical) density variations, including profiles from rather localized depth variation, to density variations with some vertical extension. We compare the velocity profiles due to solitary waves with approximately corresponding amplitudes obtained by computations and experiments. We find very good agreement between the different methods, see figure 2. In this example $h_1/h_2 = 4$ and $|Y|_{max}/h_2 = 0.68$. This means that the interface method may be applied also to a stratified fluid, as long as a typical wave is much longer than the thickness of the stratification.

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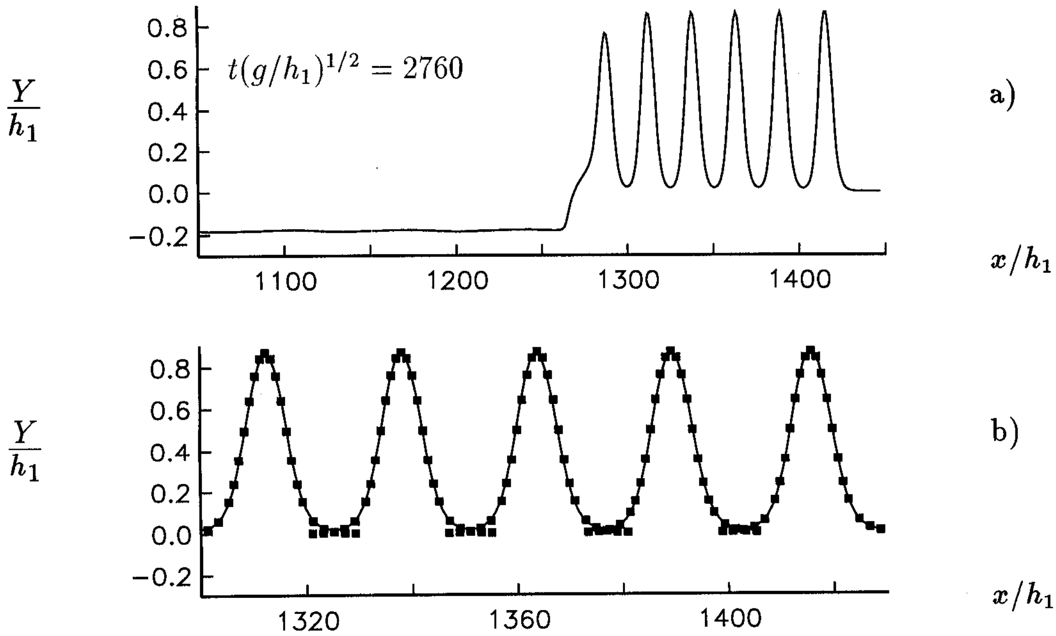

Figure 1: Generation of upstream solitary waves. Moving elliptical bottom topography with major half-axis (horizontal) $10h_1$, minor half-axis (vertical) $0.1h_1$. $U/c_0 = 1.1$, $\mu = 0.7873$, $h_2/h_1 = 4$. (a) Profile after $t\sqrt{g/h_1} = 2760$. (b) Close up of figure (a), black squares mark steady solitary wave solution with $|Y|_{max}/h_1 = 0.869$.

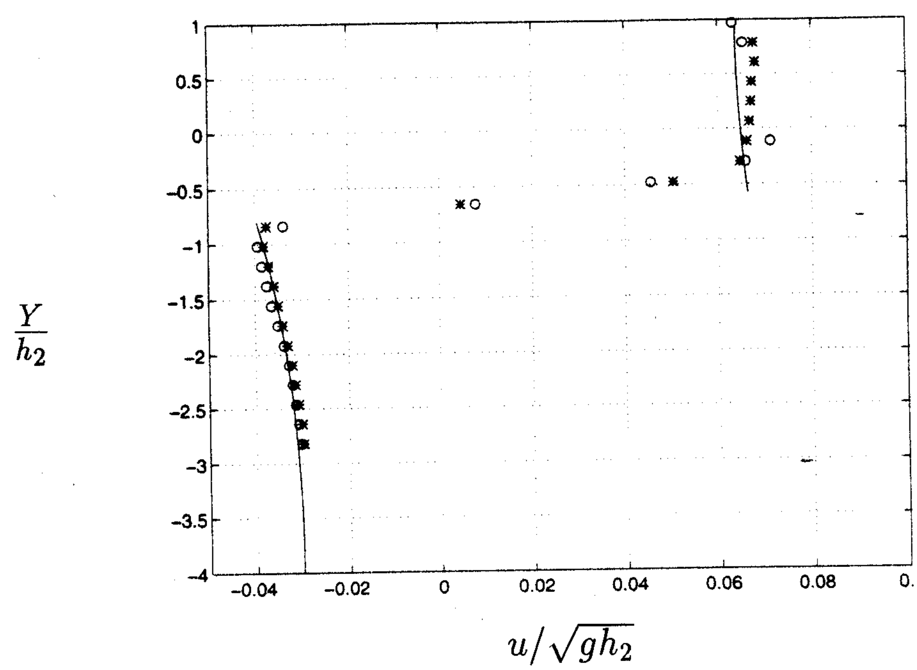

Figure 2: Velocity profile at the crest of a solitary wave. $*$ and $\circ$: experiments with different stratification. Solid line: two-layer model (theory). $|Y|_{max}/h_2 = 0.68$, $\mu = 0.978$, $h_1/h_2 = 4$.

DISCUSSION

Grilli S.: What type of boundary conditions did you use on lateral boundaries of your model? Did you translate the model with the mean wave velocity?

Grue J., Palm E.: The lateral boundaries are assumed far away such that the flow there may be considered to be zero. The model is translated with the mean wave velocity–approximately–in the computations.