Edge Waves in a Two-Layer Fluid

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1 Introduction

The goal of this presentation is the study of edge waves in a two-layer fluid. No exact solutions similar to those of Ursell [U] (except for the Stokes mode [K1]) are known in this case even for a bottom of constant slope. Trapped waves in a two-layer fluid were studied, as far as the present author knows, only in two papers [K2, K3], but under the assumptions that the interface does not intersect the bottom or obstacles—thus no internal edge waves. Therefore the construction of some analogs of trapped edge waves for a two-layer fluid with an interface not coinciding with the whole plane is of interest.

We note that, heuristically, we could expect the existence of trapped waves even in a situation when a one-layer fluid does not have them. More precisely, consider a two-layer fluid with the free surface coinciding with the plane $y = 0$ whose depth H depends only on one horizontal coordinate (say, x), and is a smoothed step, that is, H increases monotonically and $H(x) \to H_0$ as $x \to -\infty$, $H(x) \to H_1$ as $x \to \infty$, $H_0 < H_1$. Suppose further that the interface $y = -b$ of the layers is such that $H(0) = b$ (the line $x = 0, y = -b$ plays the role of a shore for the lower layer). If the densities ρ_1 and ρ_2 are equal, there are no trapped waves. On the other hand, if the density ratio $\beta = \rho_1/\rho_2$ is very small, we are "almost" in the situation of edge waves in the lower layer only, and these should exist independently of what happens in the upper layer. It turns out that these heuristic considerations can be made fairly rigorous in the case when H varies slowly. Moreover, trapped edge waves exist for *any* density ratio provided that the slope is sufficiently small. This seems to be natural in view of the properties of Ursell's modes whose total number tends to infinity as the slope tends to zero; some of them survive even when the density of the upper layer is not vanishingly small.

The proof of these results goes along the following lines. We construct explicitly approximate edge-wave solutions of the corresponding system of equations for certain values of the frequency, prove that these values lie outside the continuous spectrum, and apply a standard argument [MF] which ensures that our approximate eigenfrequencies do in fact provide asymptotics of the point spectrum of the problem. This scheme involves a reduction of the initial system to a standard form $K\phi = \omega^2\phi$ with a self-adjoint operator K; the corresponding considerations were already carried out in [ZI]. The harder part of the explicit construction of approximate eigenfunctions follows closely [Mi, Z1, Z2]: we reduce the initial system to an integral equation on the bottom and apply the WKB method (more precisely, its modification in the sense of [MF, Z1]). Seemingly, this approach can be used for a variety of different topographies as is done for a homogeneous fluid in [SMK]; here we shall restrict ourselves to the geometry described above.

2 Construction of approximate eigenfunctions

We look for waves of the form $\Phi(x, y)\exp\{i(kz - \omega t)\}$, and, in order to meet the requirement that H varies slowly, shall assume that $kH(x) = h(\epsilon kx)$, where $\epsilon \ll 1$ (we take $\epsilon = H'(0)$) and $h(x)$ is a smooth

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function analytic in a neighborhood of the origin; moreover, $h'(0) = 1$. Passing to dimensionless variables $\epsilon kx \to x$, $ky \to y$, we have for the potentials $\Phi^{(1,2)}$ of the upper and lower layers, respectively, the following system of equations and boundary conditions:

$$\Phi_y^{(1)} = \lambda^2 \Phi^{(1)}, \qquad y = 0, \tag{1}$$
$$\Phi_{yy}^{(1)} + \epsilon^2 \Phi_{xx}^{(1)} = \Phi^{(1)}, \qquad x, y \in \Omega_1, \tag{2}$$
$$\Phi_y^{(1)} + \epsilon^2 h' \Phi_y^{(1)} = 0, \qquad y = -h(x), \quad x < 0, \tag{3}$$
$$\Phi_y^{(1)} = \Phi_y^{(2)}, \qquad y = -c, \quad x > 0, \tag{4}$$
$$\beta(\Phi_y^{(1)} - \lambda^2 \Phi^{(1)}) = \Phi_y^{(2)} - \lambda^2 \Phi^{(2)}, \qquad y = -c, \quad x > 0, \tag{5}$$
$$\Phi_{yy}^{(2)} + \epsilon^2 \Phi_{xx}^{(2)} = \Phi^{(2)}, \qquad x, y \in \Omega_2, \tag{6}$$
$$\Phi_y^{(2)} + \epsilon^2 h' \Phi_x^{(2)} = 0, \qquad y = -h(x), \quad x > 0; \tag{7}$$

here $c = kb$, $\Omega_1 = \{0 < y < \min\{h(x), c\}, -\infty < x < \infty\}$, $\Omega_2 = \{-c < y < -h(x), x > 0\}$ (Ω_1 contains the fluid of the upper layer, and Ω_2 that of the lower one), $\lambda^2 = \omega^2 (gk)^{-1}$. We shall assume that $\beta \in [0, 1)$ is a fixed number independent of ϵ.

Following [Mi, Z1], we reduce the system (1)–(7) to one integral equation. To this end, we look for $\Phi^{(1,2)}$ in the form

$$\Phi^{(1,2)} = \frac{1}{\sqrt{2\pi\epsilon}} \int e^{ipx/\epsilon} (A_{1,2}(p) \cosh \kappa y + B_{1,2}(p) \sinh \kappa y)\, dp, \qquad \kappa = \sqrt{1 + p^2}, \tag{8}$$

where the integration is carried out along a contour C to be specified later and $A_{1,2}, B_{1,2}$ are new unknown functions. The integrals (8) satisfy the reduced wave equations in $\Omega_{1,2}$ exactly. In order to fulfil the conditions on the free surface $y = 0$ and the interface $y = -c, x > 0$, we choose B_1, A_2, B_2 in the following way:

$$\begin{aligned} B_1 &= A_1 \lambda^2 / \kappa \equiv A_1 M_1, \\ A_2 &= A_1 (1 + \alpha\mu \sinh \kappa c \cosh \kappa c) \equiv A_1 M_2, \\ B_2 &= A_1 (\lambda^2 / \kappa + \alpha\mu \sinh^2 \kappa c) \equiv A_1 M_3; \end{aligned}$$

here $\mu = \lambda^2/\kappa - \kappa/\lambda^2$, $\alpha = 1 - \beta$. Straightforward calculations show that under such choice of $A_2, B_{1,2}$, the integrals (8) satisfy the conditions mentioned above exactly. Now substituting $\Phi^{(1,2)}$ in (3) and (7), we obtain the following integral equation for A_1:

$$\int e^{ipx/\epsilon} L(x, p, \lambda, \epsilon) A_1(p)\, dp = 0, \tag{9}$$

where

$$L(x, p, \lambda, \epsilon) = \begin{cases} M_1 \kappa \cosh \kappa h - \kappa \sinh \kappa h + i\epsilon p h' (\cosh \kappa h - M_1 \sinh \kappa h), & x < 0, \\ M_3 \kappa \cosh \kappa h - M_2 \kappa \sinh \kappa h + i\epsilon p h' (M_2 \cosh \kappa h - M_3 \sinh \kappa h), & x > 0. \end{cases}$$

Thus our problem is reduced to finding approximate solutions of this equation. Similarly to [Z1, Z2], we look for A_1 and λ^2 in the form

$$A_1 = a(p, \epsilon) \exp(-iS(p)/\epsilon), \qquad a(p, \epsilon) = a_0(p) + \epsilon a_1(p) + \ldots, \qquad \lambda^2 = \lambda_0^2 + \epsilon^2 \lambda_1^2 + \ldots$$

(the fact that the term with the first power of ϵ vanishes in the expansion for λ^2 is a corollary of calculations). The asymptotics of the integral in (9) can be calculated, as in [Z1], according to the ideas of the stationary phase method. In the leading order, this yields an equation for S:

$$L(q, p, \lambda_0, 0) = 0, \qquad q = S_p. \tag{10}$$

This in fact is two equations (for $q < 0$ and $q > 0$). It turns out that for sufficiently small λ there exist solutions with $q > 0$ and none with $q < 0$, which, as in the case of a one-layer fluid, have the specific form $q \sim \lambda_0^2/\alpha(1+p^2)$ and describe trajectories of the corresponding Hamiltonian system escaping to infinity in p as $q \to 0$ (cf. [Z1, Z3]). Indeed, (10) for $q < 0$ reads

$$\lambda_0^2 = \kappa \tanh \kappa h(q);$$

if $\lambda_0^2 < \tanh h_0$ (here $h_0 = kH_0$), then there are no solutions. For $q > 0$, after some elementary manipulations, we obtain

$$\tanh \kappa(h(q) - c) = \frac{\lambda_0^2}{\kappa f(\kappa, \lambda_0)}, \qquad f(\kappa, \lambda) = 1 - \frac{\beta(\lambda^4 - \kappa^2)\kappa^{-1} \tanh \kappa c}{\lambda^2 - \kappa \tanh \kappa c}. \tag{11}$$

The factor $f(\kappa, \lambda)$ satisfies

$$f(\kappa, \lambda) = \alpha + O(\lambda^2/\kappa) \qquad \text{as} \qquad \kappa \to \infty, \quad \lambda \to 0,$$

and never vanishes for sufficiently small λ. Since $h(0) = c$ and $h'(0) = 1$, the solution, by the implicit function theorem, has the form

$$q(p, \lambda_0) = \frac{\lambda_0^2}{\alpha\kappa^2} + u(\lambda_0^2\kappa^{-2}, \lambda_0^2(\kappa \tanh \kappa c)^{-1}, \lambda_0^2),$$

where u is analytic in its arguments and has zero linear part in a neighborhood of the origin. Thus $S_p = q$ is analytic in p in some sectors with apex at the origin and containing the real axis.

The equations for a_j, $j \geq 0$, are obtained by a word for word repetition of the arguments from [Z1]; it turns out that

$$a_0 = \kappa^{-1} v(\lambda_0^2\kappa^{-2}, \lambda_0^2(\kappa \tanh \kappa c)^{-1}, \lambda_0^2)$$

with an analytic v.

By the analytic properties of the phase and amplitude, it is possible to take as the contour of the integration in (8) the real axis deformed into the upper half-plane for $x > 0$ and into the lower half-plane for $x < 0$. This ensures the convergence of the corresponding integrals. Now the conditions that $\Phi^{(1,2)}$ do not have singularities at $x = 0$ lead to

$$\frac{1}{\epsilon} \int_{-\infty}^{\infty} q(p, \lambda_0)\, dp = (2n+1)\pi, \qquad n = 0, 1, \ldots, \tag{12}$$

which defines a discrete set of eigenfrequencies. The corresponding integrals (8) now define functions which oscillate in the interval $x \subset (0, q(0, \lambda_0))$ and decay exponentially outside this interval. The asymptotics of these functions can be obtained as in [Z2, Z3] in terms of Laguerre functions. We note that the procedure of [Z2] cannot be used directly to estimate the residual terms because, in contrast to [Z2], $q(p, \lambda_0)$ is not analytic in p in a full neighborhood of infinity. Nevertheless, if in the expression for f in (11) one puts $\tanh \kappa c \equiv 1$, then q is analytic and differs from the original q by $O(\kappa^{-\infty})$. This fact enables one to apply the scheme of [Z2] and prove that (9) is satisfied up to a small in ϵ term.

For small numbers n formula (12) means that λ_0 is small, and it is not hard to see (using the analytic form of q), that for such n (12) reduces to

$$\lambda_0^2 = \alpha\epsilon(2n+1) + O(\epsilon^2), \tag{13}$$

which in the case $\alpha = 1$ (the upper layer is of density 0) is a limiting form of Ursell's result.

The continuous spectrum of our problem is given by

$$\lambda^2 \geq \Lambda^2 = \min\{\tanh h_0, \Lambda_1^2\},$$

where Λ_1^2 is the lower bound of the spectrum of the two-layer problem with the interface at $y = -c$ and the (straight) bottom at $y = -h_1 \equiv kH_1$. It is well-known that Λ^2 is strictly positive (although small for small α or h_0). Thus the values (13) for sufficiently small ϵ and n *always* lie outside the continuous spectrum.

We note that for sufficiently small α or h_0 our eigenvalues can, for n large, become embedded in the continuous spectrum. It seems reasonable to conjecture that in this case they correspond to complex eigenvalues with an exponentially small in ϵ imaginary part (as in [LM]). Our approach of asymptotics in powers of ϵ cannot provide any information on this quantity.

3 Conclusions

We have shown that for a bottom topography which excludes the existence of trapped modes in a one-layer fluid, there exist analogs of Ursell's edge waves when the bottom plays the role of a shore for the lower layer. It is shown further that such modes exist for any density ratio of the layers provided the slope of the bottom is sufficiently small. Finally, for small mode numbers an extremely simple formula (13) for the frequencies is obtained.

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