Lifting flow about a smooth contour moving beneath the free surface

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1 Introduction

The flow with non-zero circulation about a two-dimensional body with continous slope moving steadily under the free surface of non-viscous fluid is investigated. The salient feature of the present paper is that the circulation is not assumed to be arbitrary but is determined with a help of some Kutta-like heuristic condition (the original Kutta condition is not applicable but to contours with sharp edges or corners). The idea of extension of Kutta condition to completely smooth profiles which *do* generate lift was inspired by Saren (1975) who had proposed basing on the Hamilton's principle to chose the circulation in such a way that the average squared slip velocity around the *whole* contour be minimized. It is obvious that for any profile with a unique convex corner that condition becomes identical to the standard Kutta condition. Unfortunately, the Saren's condition leads to wrong results for any profile having two axes of symmetry including flat plate. This drawback has been removed through modification of Saren's condition proposed by the author (Sutulo (1988)). The modification was quite simple: velocity averageing is supposed to be carried out along *some rear part* of the profile. This modified condition was named *non-localized variational condition (NLVC)* and its application to the problem of lift on an elliptical contour in the unbounded non-viscous fluid brought unexpectedly interesting results providing more realistic estimates of lift gradient on profile with *slightly* rounded trailing edge than the Kutta condition does. Somewhat unpleasant is arbitrariness of the mentioned "rear part " of contour (we have considered all the tail region starting from the midchord section): it is supposed that that should be the part where the viscosity affects the flow in some way. Though, after some reasonning the author came to conclusion this arbitrariness was natural for a heuristic condition and that it was in fact even less pronounced than in the case of the original Kutta condition when *a specific point* is to be indicated instead of some *region.*

A relative success of the first application of the variational condition encouraged the author to undertake an attempt to apply the NLVC to a more general case of arbitrary smooth profile moving under the free surface at finite Froude numbers.

2 Statement of problem

Let Oxy be a Cartesian two-dimensional coordinate system with x-axis lying on the undisturbed free surface and y-axis directed upward. A contour (profile) L is fixed somewhere beneath the free surface and all the points on the contour have coordinates (ξ, η). The contour is subject to action of a uniform stream having velocity $\vec{V}_\infty$ collinear to x-axis. The resulting flow is supposed to have potential

$$\Phi = V_\infty x + \varphi(x, y)$$

where φ is disturbance potential satisfying the following boundary value problem:

- Laplace equation

$$\Delta\varphi = 0 \quad \text{at} \quad y < 0;$$

- slip condition on the contour

$$\frac{\partial\varphi}{\partial n} = -\vec{V}_\infty \cdot \vec{n}$$

where $\vec{n}$ is unity outer normal vector;

- linearized free-surface boundary condition

$$\frac{\partial^2\varphi}{\partial x^2} - \nu\frac{\partial\varphi}{\partial y} = 0 \quad \text{at} \quad y = 0$$

where $\nu = g/V_\infty^2$; g being acceleration of gravity;

- conditions at infinity

$$\vec{v} = \nabla\varphi \to 0 \quad \text{at} \quad y \to -\infty \quad \text{or} \quad x \to -\infty.$$

Naturally, this formulation does not define the velocity field uniqually as some circulation flow can be added which does not violate any boundary condition. Absolute fluid velocity at arbitrary point of fluid domain can then be represented as follows:

$$\vec{V} = \vec{V}_\infty + \vec{v}_q + \Gamma\vec{v}_\Gamma \tag{1}$$

where $\vec{v}_q$ is disturbance velocity at zero circulation; Γ is circulation value and $\vec{v}_\Gamma$ is disturbance velocity due to unity circulation.

The circulation is to be determined from the NLVC which can be formulated as

$$\int\limits_{L_0} V^2 dL = \min \qquad \Longleftrightarrow \qquad \frac{\partial}{\partial\Gamma}\int\limits_{L_0} V^2 dL = 0. \tag{2}$$

3 Main relations

The following boundary integral equation for the unknown source density $q(P)$ is in effect

$$\pi q(M) + \vec{n}_M \cdot \nabla \int\limits_L q(P) G_q(M,P) dL = -\vec{n}_M \cdot (\vec{V}_\infty + \Gamma\vec{v}_\Gamma) \tag{3}$$

where $M(x,y)$ and $P(\xi,\eta)$ are points (here: lying on the contour); $G_q()$ is point source Green function and

$$\vec{v}_\Gamma = \frac{1}{L}\nabla\int\limits_L G_\Gamma(M,P)dL$$

where $G_\Gamma()$ is Green function for a point vortex (it is clear that a uniform vorticity distribution is taken along the contour).

The equation (3) can be used for evaluation of source density $q(M)$ while an additional equation is necessary for determination of circulation Γ. This can be obtained in the explicit form by taking into account (2) and (1)

$$\Gamma = \frac{-\int\limits_{L_0} \vec{v}_\Gamma \cdot (\vec{V}_\infty + \vec{v}_q) dL}{\int\limits_{L_0} v_\Gamma^2 dL}. \tag{4}$$

The latter equation can be substituted into (3) resulting in a *nonlinear* integral equation for source density. In fact, it is better to solve (3) iteratively evaluating Γ at each step with a help of (4).

Any induced velocity $\vec{v} = v_x\vec{e}_x + v_y\vec{e}_y$ where $\vec{e}_{x,y}$ are coordinate vectors and velocity components can be represented as components of complex velocity $\frac{dw}{dz}$ depending on complex variable $z = x + iy$:

$$v_{q,\Gamma x} = \Re(\frac{dw_{q,\Gamma}}{dz}) \qquad v_{q,\Gamma y} = -\Im(\frac{dw_{q,\Gamma}}{dz}).$$

The complex velocities are

$$\frac{dw_q}{dz} = \int\limits_L q(\zeta)\hat{G}_q(z,\zeta)dL \qquad \frac{dw_\Gamma}{dz} = \frac{1}{L}\int\limits_L \hat{G}_\Gamma(z,\zeta)dL$$

where complex Green functions are introduced (another complex space variable $\zeta = \xi + i\eta$ and asterisks are used for complex conjugates):

$$\hat{G}_q(z,\zeta) = \frac{1}{z-\zeta} + \frac{1}{z-\zeta^*} - 2\pi\nu e^{-i\nu(z-\zeta^*)} - 2i\ v.p.\int\limits_0^\infty \frac{ke^{-ik(z-\zeta^*)}}{k-\nu}dk;$$
$$\hat{G}_\Gamma(z,\zeta) = i[\frac{2}{z-\zeta} - \hat{G}_q(z,\zeta)].$$

When source density distribution and circulation value are known, lift, drag and moment acting upon the contour are found through integration of pressure distribution obtained from the Bernoulli equation. Then, say, the total lift coefficient C_L can be represented as follows:

$$C_L = C_{Lq} + C_{L\Gamma} + C_{L\Gamma\Gamma}$$

where

$$C_{Lq} = -\frac{1}{\ell}\int\limits_L (\bar{v}_q^2 + 2\bar{v}_{qx})dx; \quad C_{L\Gamma} = -\frac{2\tilde{\Gamma}}{\ell}\int\limits_L (\bar{\vec{v}}_q \cdot \bar{\vec{v}}_\Gamma + \bar{v}_{\Gamma x})dx; \quad C_{L\Gamma\Gamma} = -\frac{\tilde{\Gamma}^2}{\ell}\int\limits_L \bar{v}_\Gamma^2 dx.$$

where bars and tildes denote dimensionless quantities (based on V_∞ and on the contour's chord ℓ).

A numerical algorithm has been built in a more or less standard manner (see e.g. Hess & Smith (1967)): the contour was approximated with an inscribed polygon having constant source density distributed on each segment. On the other hand, the algorithm was implemented on the basis of concepts of object-oriented programming: a number of special C++ classes were created (`TwoDimGeomHS, TwoDimSourceVortHS etc.`) and standard classes of vectors and matrices were used.

Numerical results of calculations of lift and drag will be presented for elliptical and some other smooth profiles moving at various submergencies and Froude numbers.

References

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