Solitary wave splitting due to a mildly sloping bottom

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Abstract

The distortion of a travelling solitary wave under the influence of a slowly decreasing water depth is investigated numerically. Simulations with a time domain boundary element method for nonlinear gravity waves indicate that the wave steepens, becomes slower and splits into two waves of solitary shape travelling at different velocities.

1 Introduction

In this paper we focus on the distortion of a solitary wave due to a slowly decreasing water depth. This research was motivated by recent theoretical investigations of Van Groesen & Pudjaprasetya[2,8], based on a modified version of the well-known Korteweg-de Vries (KdV) equation[5]. This theory predicts the transition of a 1-soliton into a 2-soliton under the influence of a mildly sloping bottom; the numerical calculations reported here seem to indicate that this effect occurs indeed. The effect of an uneven bottom on this special type of water waves is obviously of engineering importance; the occurrence of analogous situations in widely different physical phenomena (e.g. in optics: irregularities in glass fibre cables) adds even more practical relevance to this topic. Earlier work in this specific area of interest is due to (for instance) Madsen & Mei[6], Miles[7], Knickerbocker & Newell[4] and, very recently, Johnson[3]. However, in none of these publications there seems to be clear evidence of *exact splitting* of solitary waves, neither from theoretical and numerical investigations nor from experimental observations.

The outline of this paper is as follows: first we give a concise description of the mathematical model for two-dimensional nonlinear free surface potential flow under the action of gravity. Next, we briefly discuss the main features of the time domain boundary element method (BEM) that we developed to simulate nonlinear gravity waves. Then we present a short survey of the theoretical investigations of Van Groesen & Pudjaprasetya. Preliminary numerical computations clearly indicate that, at least in a qualitative sense, the effect of a decreasing water depth on a solitary wave is fourfold: the amplitude *in*creases and the wavelength *de*creases (i.e. the wave steepens), the velocity of the main wave *de*creases and a second (smaller) wave splits off at a lower speed. In the end, this research aims at a quantitative and qualitative comparison of theoretical results (for simplified equations like Boussinesq, and possibly with KdV-type of equations provided it can be shown that effects of reflection at the bottom are negligible), BEM-numerics and small scale experiments with regard to the deformation of a solitary wave over an uneven bottom.

2 Mathematical model and computational method

Consider the evolution of waves travelling on the surface of an ideal fluid under the action of gravity alone, assuming the flow to be free of rotation. This so-called classical water wave problem in its full-dimensional form is described by the following set of equations:

$$\Delta\phi \equiv \nabla^2\phi = 0 \qquad \text{in } \Omega \ , \tag{1}$$

$$\left.\begin{aligned} \dot{\phi} &= \tfrac{1}{2}\nabla\phi\cdot\nabla\phi - gz \\ \dot{\eta} &= \eta_t + \phi_x\eta_x = \phi_z \end{aligned}\right\} \qquad \text{at } z = \eta(x;t) \ , \tag{2}$$

$$\phi_n = \nabla\phi\cdot\vec{n} = 0 \qquad \text{at } z = -h(x) \ . \tag{3}$$

In these expressions, the dots denote material derivatives (i.e. following the motion of a water particle) and the subscripts denote partial derivatives. As usual, g is the gravitational acceleration and $\vec{n}$ is the unit normal vector on the boundary $\partial\Omega$. The velocity potential ϕ is introduced under the aforementioned assumption. Eq. 1 is the continuity equation which is valid throughout the fluid domain Ω; Eq. 2 contains the dynamic and kinematic conditions at the free surface, describing the wave motion in terms of the potential and the elevation η; finally, the impermeability of the bottom is expressed in Eq. 3; see Figure 1 for the geometrical definitions.

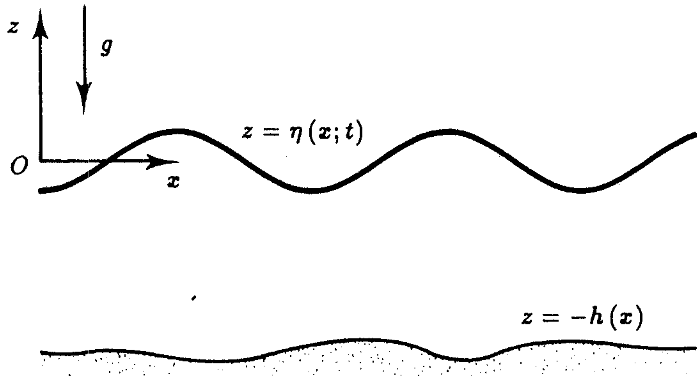

Figure 1: Definition of geometry.

Since the time-dependence comes in through the free surface conditions (Eqs. 2) only, this problem can be split into two subproblems which are solved step by step. The time-independent part is governed by Laplace's equation (Eq. 1) which, using Green's identity, is transformed into a boundary integral equation (BIE):

$$\frac{1}{2}\phi(\vec{x}) = \int_{\partial\Omega} \left[\frac{\partial\phi}{\partial n_\xi}\left(\vec{\xi}\right) G\left(||\vec{x}-\vec{\xi}\,||\right) - \phi\left(\vec{\xi}\right)\frac{\partial G}{\partial n_\xi}\left(||\vec{x}-\vec{\xi}\,||\right)\right] dS_\xi \ , \tag{4}$$

where G is the Green's function ($G(r) = \frac{1}{2\pi}\ln\frac{1}{r}$ in 2D) and integration is over the boundary of the fluid domain Ω. In this continuous form, this BIE is applied in each point $\vec{x}$ on $\partial\Omega$. In a discretized form the boundary is approximated by a finite number of boundary elements, each represented by (in our approach) one collocation point $\vec{x}_i$ situated in the middle of the element. The BIE is applied in each point $\vec{x}_i$, so that a system of linear equations is obtained:

$$\frac{1}{2}\phi^i = \sum_{j=1}^{N}\left[C_s^{ij}\phi_n^j + C_d^{ij}\phi^j\right] \ , \tag{5}$$

where C_s^{ij} and C_d^{ij} are the source and dipole coefficients respectively and summation is over all N collocation points. Substitution of ϕ for Dirichlet boundaries and ϕ_n for Neumann boundaries yields N linear equations in exactly N unknowns, which can be solved using direct methods (e.g. Gaussian elimination) or iterative methods (e.g. conjugate gradients type). The solution contains ϕ_n for the Dirichlet boundaries and ϕ for the Neumann boundaries.

Next we have to solve the time dependent part of the problem, especially for the evolution of the free surface. The new positions of the collocation points are determined by integrating the kinematic conditions in time. The new values of the potential at the free surface collocation points are obtained by integrating the dynamic condition. For the time marching we use a fourth order classical Runge-Kutta scheme, which implies that the BVP has to be solved on four levels for each time step. For a detailed description of the mathematical model and the computational method we refer to Van Daalen[1].

3 Modified KdV-equation for mildly sloping bottom

For gravity driven surface waves Van Groesen & Pudjaprasetya [2] derived the governing equation for waves travelling mainly in one direction. Assuming mild bottom variations and rather long and low waves, they obtained a modified KdV-equation with coefficients depending on the bottom topography:

$$\partial_t\eta = -\Gamma(x)\,\delta_\eta H(\eta) \ , \tag{6}$$

with

$$\Gamma(x) = \frac{1}{2}\left[c(x)\,\partial_x + \partial_x c(x)\right] \text{ and } c(x) = \sqrt{gh(x)} \ , \tag{7}$$

and the total energy as the Hamiltonian:

$$H(\eta) = \int \left[\frac{1}{2}\eta^2 + \epsilon\left(-\frac{1}{12}h^2(x)\eta_x^2 + \frac{\eta^3}{4h(x)}\right)\right] dx \ . \tag{8}$$

When $h(x)$ is replaced by h_0 (constant), Eq. (6) reduces to the familiar KdV-equation:

$$\partial_t \eta = -c_0 \partial_x \delta_\eta H_0(\eta) \ \text{ with } \ c_0 = \sqrt{gh_0} \ , \tag{9}$$

where

$$H_0(\eta) = \int \left[\frac{1}{2}\eta^2 + \epsilon\left(-\frac{1}{12}h_0^2\eta_x^2 + \frac{\eta^3}{4h_0}\right)\right] dx \tag{10}$$

is constant during the evolution. This standard KdV-equation admits steady travelling waves propagating at constant speed. In case of an uneven bottom however, there is no translation symmetry and the solitary wave will distort. During run-up the amplitude of the wave will increase and its wavelength and velocity will decrease; at the back of the wave there appears a tail, i.e. a constant function (in space) which has compact support. The KdV-equation is well known as a completely integrable system, having N-soliton solutions ($N \geq 1$) on the whole real line [10], which are the solutions of the extremum problem (for the energy) with one or more constraints. Pudjaprasetya & Van Groesen have also studied the transition of a one-soliton into a two-soliton during run-up, including a bifurcation analysis. Their research motivated the computations described hereafter.

4 First results on soliton splitting due to a mildly sloping bottom

A two-dimensional steady periodic solution of the nonlinear water wave problem (Eqs. 1-3) with a horizontal bottom, — i.e. $h(x) = h_0$ — was proposed in the form of Fourier series for the potential and the elevation by Rienecker & Fenton[9]. Our strategy is to approximate a solitary wave by taking the wavelength ($\lambda = 2\pi/k$) in this solution extremely large with respect to the wave height h, such that the wave crests are confined to a relatively small region, where the wave troughs are very long. A solitary wave profile is approximated by 32-term (convergent) Fourier series for the elevation and potential corresponding to a steady periodic wave with 0.15m amplitude and 40m wavelength ($A : \lambda \approx 1 : 267$) on 0.50m deep water ($A : h \approx 1 : 3$). The effect of soliton splitting was first seen in the following configuration: the tank length is 40m, and the water depth decreases linearly from $h = 0.50$m to $h = 0.35$m between $x = 10$m and $x = 15$m (i.e. a 1:33 slope). The center of the wave is initially located at $x_0 = 5$m, it travels to the right at speed $v \approx 2.47$m/s. The panel distribution and the time step are the same as in the previous simulation. Figure 2 shows the bottom topography and the wave profiles at subsequent stages in the evolution ($t = 0, 3, \ldots, 15$s). It can be observed that the effect of the sloping bottom on the soliton is in some sense rather complex and in another sense very simple: in the process of deformation, the wave amplitude increases and the wavelength decreases, i.e. the wave steepens; moreover, its speed reduces significantly. More surprising and fascinating however, is the splitting of the wave into two waves which, at a first glance, are of solitary type. The positions of the wave crests are also indicated in this plot: it is seen that the velocity decreases once the wave has passed the sloping region (see the splitted line: the lower part denotes the undisturbed velocity and the upper part corresponds to the disturbed wave) and that the second crest (which can be detected only after 7.5s, this is the short line) travels at a lower speed than the first crest. Other calculations involving different (both milder and steeper) slopes show the same effects on the solitary wave; more detailed results will be published elsewhere.

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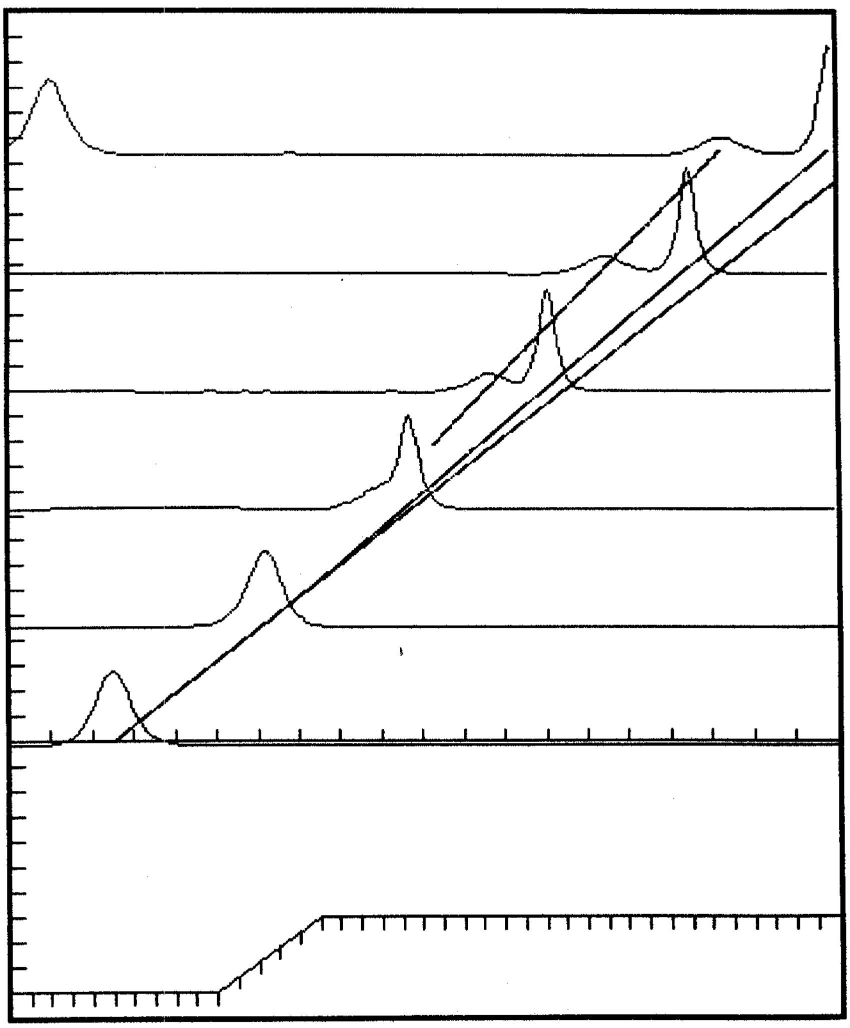

Figure 2: Evolution of a solitary wave over a mildly sloping bottom.