# Hydrodynamic loads on a cylinder moving unsteadily in a 3-D non-uniform flow field

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## Abstract

In this paper we present a direct method for calculating the hydrodynamic loads (forces and moments) acting on a rigid cylinder (of arbitrary cross-section) moving unsteadily in a direction perpendicular to its main axis and rotating about the same axis in a *non-uniform* ambient potential flow field $\mathbf{V}(\mathbf{x}, t) = \nabla\phi$. The corresponding expressions for the force and moment are given in a moving (body-fixed) coordinate system and the derivation is based on the general methodology recently developed by Galper & Miloh (1995). The presence of a free-surface is indirectly accounted for through a self-consistent correction for the ambient stream $\mathbf{V}$ (see, for example, Faltinsen et al. 1995). It is further implied that the axis of the cylinder coincides with the $z$-axis of the corresponding cylindrical coordinate system.

We consider firstly the particular example of an infinitely long (in the $z$-direction) cylinder, the diameter $d$ of which is much smaller than the length scale $L$ of the $z$-non-uniformity of the ambient flow field. No restrictions are imposed on the flow non-uniformity in the $(x, y)$ plane. In this case, up to the zero-order of the small parameter $\epsilon \equiv \frac{d}{L} \ll 1$, the problem is globally a 2-dimensional one. Nevertheless, locally the problem preserves its 3-D nature, which manifests itself in the fact that the local force acting on any particular cross-section is given by the sum of a purely 2-D force (per unit length) plus an additional force per unit length $\mathbf{F}^a$ proportional to the fluid acceleration in the $z$-direction, i.e.,

$$\mathbf{F}^a = \frac{1}{2}\int_S (V_z^2)(\mathbf{r}, z)\, n_i\, dS, \tag{1}$$

where $\mathbf{V} \equiv (V_x, V_y, V_z) = (V_1, V_2, V_3)$ and $S(\mathbf{r}) = 0$ is the equation of a cross-section. Assuming that the ends of the cylinder are in a fluid at rest, the force per unit length is determined up to a full $z$-derivative (which clearly vanishes after the $z$-integration). The last term in the RHS of (1) can be written after subtracting the full $z$-derivative as,

$$\frac{1}{2}\int_S (V_3^2)(\mathbf{r}, z)\, n_i\, dS = \frac{1}{2}\int_S \phi(\mathbf{r}, z)\, E_{33}(\mathbf{r}, z)\, n_i\, dS, \tag{2}$$

from where one concludes that in order to reduce the problem to a pure 2-D forces, the form of the strain tensor should be degenerated, i.e.

$$E_{33} = 0, \quad or \quad E_{3i} = 0, \quad i = 1, 2, \tag{3}$$

where $E_{ij} \equiv \nabla_i V_j$.

The derived expressions, when applied for the particular case of a weakly non-uniform flow field (in the transverse plane) coincide, with the common expressions for a force per unit length $\tilde{\mathbf{F}}$ acting on a *moving* cylinder with a velocity $\mathbf{U}$ in $x, y$-plane (see, for example, Rainey 1995, Dowling 1994, Manners & Rainey 1992 and Madsen 1986)

$$\tilde{\mathbf{F}} = (s\hat{1} + \hat{m})\frac{D\mathbf{V}}{Dt} + (\hat{E}\hat{M} - \hat{M}\hat{E})(\mathbf{V} - \mathbf{U}), \tag{4}$$

where $s$ is the area of the cylindrical cross-section, $\hat{M}$ is the 3-D translational added-mass tensor and $\hat{m}$ is the 2-D added-mass tensor of the cross-section. Note that for a circular cylinder this force can be expressed in a potential form.

We consider further the next-order terms in the theory of perturbations, namely those of $O(\epsilon^2)$. The corresponding expressions suggest now the existence of *long-distance* coupling between different cross-sections. Let us write the $2-D$ shape $S$ in a complex plane as a Fourier expansion

$$\zeta = r^* e^{i\psi} + \sum_{n=1} c_n \exp(-in\psi), \tag{5}$$

where $r^*$ is a "conformal radius" given by

$$4\pi r^{*2} = Tr(\hat{m}) + 2s, \tag{6}$$

and $Tr(\hat{m})$ denotes the trace of the added-mass matrix $\hat{m}$. Note that with the help of the corresponding rotation of the plane, one can always choose the coordinate system in such a way that $Im(c_1) = 0$ (in this coordinate system the $2-D$ added-mass tensor $\hat{m}$ is diagonalized). The force per unit length, acting on a non-circular cross-section embedded in a weakly non-uniform flow field, is given up to $O(\epsilon^2)$ by

$$\tilde{\mathbf{F}} = (s\hat{1} + \hat{m})\frac{D\mathbf{V}}{Dt} + (\hat{E}\hat{M} - \hat{M}\hat{E})\mathbf{V} + \frac{1}{2}s^* r^{*2} \hat{A} \frac{\partial^2 \hat{E}(z)}{\partial z^2} \int_{-\infty}^{\infty} \frac{\mathbf{V}(\acute{z})}{\sqrt{(z - \acute{z})^2 + r^{*2}}}\, d\acute{z}, \tag{7}$$

where we introduce a diagonal matrix

$$\hat{A} \equiv \left| \begin{array}{cc} (1+c_1)^2 & 0 \\ 0 & (1-c_1)^2 \end{array} \right|, \tag{8}$$

and denote $s^* = \pi r^{*2}$. For a circular cylinder $c_1 = 0$, $Tr(\hat{m}) = 2s = \frac{1}{2}\pi d^2$ and hence $2r^* = d$.

This coupling kernel is essentially different from the corresponding coupling one in the slender-body theory (see for example Tuck 1990) but has the same asymptotic behavior for the case of a small cross-section, i.e., $d \to 0$.

An additional force due to the rotation of the cylinder, which couples the strain tensor and the angular velocity, is also obtained. This term, which is of order $\epsilon^2$, vanishes for a symmetrical cross-section.

## References

Galper A. & Miloh T. 1995 Dynamical equations for the motion of a rigid or deformable body in an arbitrary potential non-uniform flow field. *J. Fluid Mech.* (in press).

Faltinsen, O. M., Newman J. N. & Vinje, T., 1995 Nonlinear wave loads on a slender vertical cylinder. *J. Fluid Mech.* ( in press)

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Manners, W. & Rainey, R.C.T. 1992 Hydrodynamic forces on a fixed submerged cylinders. *Proc. Roy. Soc. Lond.* A. **436**, 13.

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Tuck E. O. 1990 Analytic aspects of slender body theory. *Proc. of Ursell's Symposium.* Manchester.

# DISCUSSION

**Newman, J. N.:** I think the "non-local" force in your transparencies is not unexpected, and is essentially the same as one gets from a higher order slender body analysis. This component is essentially the force due to the (3D) dipoles coming from the outer solution. I believe it is discussed in the paper by E. O. Tuck at the Ursell Symposium (Manchester 1990).

**Galper, A., & Miloh, T.:** The same non-local term also appears in the theory of 3D vortex stretching of thin filaments with finite core, recently developed by Klein and Majda. The phrase "unexpected" has been loosely used in this context. This non-local coupling term is similar to a corresponding correction term arising in higher-order slender body theory (discussed by Tuck), however it is not the same. In slender body theory the non-local term can be obtained by a Fourier inversion of $K_0(kr)$, whereas in the present non-uniform flow analysis it is given by the inverse Fourier transform of $K_1(kr)/krK_1'(kr)$. For small $kr$ both terms are asymptotically equal to $O((kr)^2\log(kr))$.

**Rainey, R. C. T. :** As at the 1990 workshop (see 1990 workshop discussion) I greatly value this opportunity to compare my results with Prof. Miloh's. I believe he agrees with my latest, comprehensive result (1), and effectively extends it by including terms proportional to cylinder $(\text{cross-section})^2$. The long-distance coupling in these higher order terms, highlighted by Prof. Miloh, is interesting, and perhaps expected given the well-known interaction effect between a group of vertical cylinders; which is also of order $(\text{cross-section})^2$.

(1) Rainey R. C. T. (1995) Slender-body Expressions for the Wave Load on Offshore Structures. Proc. R. Soc. In press.

**Galper, A., & Miloh, T.:** The term representing interaction-effects between a group of vertical cylinders in Rainey's formulation is indeed a higher-order term proportional to $r^4$ ($r$ is the radius of the cylinder); however, it is still a <u>local</u> term unlike the <u>non-local</u> terms reported here for a single cylinder.