Mass Transport in two-dimensional wave tank

Yusaku KYOZUKA
Dept. Earth System Sci. & Tech., Kyushu University

Abstract

Mass transport induced by a wave maker in two-dimensional wave tank is studied theoretically and experimentally. Viscous effects on free-surface and water bottom are considered by the boundary layer theory on the basis of Longuet-Higgins' thoery. Contributions by local waves in front of a wave maker are exactly taken into account by the eigen function expansions. Stream lines and velocity vectors are presented graphically and they are validated in comparison with measurements in a wave tank.

1 Introduction

Mass transport in water waves is a nonlinear phenomenon and it plays important role in the environmental problems in the ocean such as the propagation of oil spills and the diffusion of contaminated materials. Stokes drift gives the steady velocity as same direction as wave propagation over the water depth, which results in contradiction of mass conservation. Therefore, Eulerian return flow is needed to satisfy the mass continuity equation. Longuet-Higgins(1953) gave analytical solutions of mass transport streaming for regular waves and partially reflected waves over a constant water depth, where the vorticity equation is solved by the laminar boundary-layer approximation at the free-surface and the water bottom. Iskandarani and Liu(1991) developed a numerical method using a spectral scheme based on a Fourier-Chebyshev expansion to compute the steady flow over a hump on the seabed.

We are interested in the control of the mass transport in waves by an ocean structure, so that we must solve the vorticity equation around an arbitrary body. A numerical method like Iskandarani and Liu may be useful generally, but a semi-analytical approach is possible for a wave maker problem in constant water depth treated here. The stream function for mass transport is calculated from the products of the first-order eigen functions for progressive wave and local waves. Some numerical results for a flap type wave maker are presented graphically and they are compared with measurements.

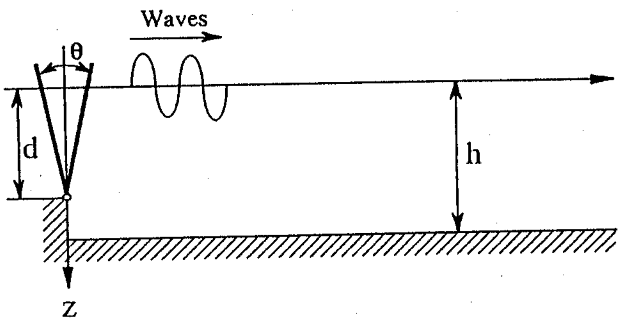

Fig.1 Coordinate system

2 Formulation of the problem

2.1 Definition of mass transport

We consider regular waves progressing on a constant water depth($z = h$) as shown in Fig.1. The velocity and the stream function are expanded in an asymptotic series of a small parameter ε, as follows.

$$\left.\begin{array}{rcl} (u, w) &=& \varepsilon(u_1, w_1) + \varepsilon^2(u_2, w_2) + \cdots, \\ \psi &=& \varepsilon\psi_1 + \varepsilon^2\psi_2 + \cdots \end{array}\right\} \quad (1)$$

Mass transport velocity($\varepsilon^2\overline{U}_2, \varepsilon^2\overline{W}_2$), defined as the average of the Lagrangian velocity, becomes

$$\left.\begin{array}{rcl} \overline{U_2} &=& \overline{u_2} + \overline{\int \frac{\partial\psi_1}{\partial z}dt\frac{\partial^2\psi_1}{\partial x\partial z}} - \overline{\int \frac{\partial\psi_1}{\partial x}dt\frac{\partial^2\psi_1}{\partial z^2}} \\ &=& \frac{\partial\Psi}{\partial z} \\ \overline{W_2} &=& \overline{w_2} - \overline{\int \frac{\partial\psi_1}{\partial z}dt\frac{\partial^2\psi_1}{\partial x^2}} + \overline{\int \frac{\partial\psi_1}{\partial x}dt\frac{\partial^2\psi_1}{\partial x\partial z}} \\ &=& -\frac{\partial\Psi}{\partial x} \end{array}\right\} \quad (2)$$

where Ψ denotes the stream function for the mass transport velocity $\overline{\mathbf{U}}$ as:

$$\Psi = \overline{\psi_2} + \overline{\int \frac{\partial\psi_1}{\partial z}dt\frac{\partial\psi_1}{\partial x}} \quad (3)$$

Stokes drift is calculated from the second-term of eq.(3).

Next, vorticity equation is obtained from the rotation of incompressible Navier-Stokes equation as:

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + w\frac{\partial}{\partial z} - \nu\nabla^2\right)\nabla^2\psi = 0 \quad (4)$$

where ν is the kinematic viscosity.

From first-order term in eq.(4), we obtain

$$\nabla^2\psi_1 = \nu\int\nabla^4\psi_1\,dt. \quad (5)$$

Second-order term of time average of eq.(4) is

$$\overline{(u_1\frac{\partial}{\partial x}+w_1\frac{\partial}{\partial z})\nabla^2\psi_1}=\nu\nabla^4\overline{\psi_2} \quad (6)$$

Substituting eq.(5) into eq.(6), we obtain

$$\nabla^4\overline{\psi_2}=\overline{(u_1\frac{\partial}{\partial x}+w_1\frac{\partial}{\partial z})\int\nabla^4\psi_1\,dt} \quad (7)$$

From eq.(3), the governing equation for Ψ becomes

$$\nabla^4\Psi=\overline{(u_1\frac{\partial}{\partial x}+w_1\frac{\partial}{\partial z})\int\nabla^4\psi_1\,dt}+\nabla^4\overline{\int\frac{\partial\psi_1}{\partial z}\,dt\frac{\partial\psi_1}{\partial x}} \quad (8)$$

This equation is treated by the boundary-layer theory. In the exterior region, eq.(8) is written as follow since we can assume ψ_1 satisfy the Laplace equation.

$$\nabla^4\Psi=\nabla^4\overline{\int\frac{\partial\psi_1}{\partial z}\,dt\frac{\partial\psi_1}{\partial x}}. \quad (9)$$

In the interior of boundary-layer, we have Stokes solution for oscillating flow on water bottom. Last result of boundary condition for Ψ on water bottom is given by Longuet-Higgins as:

$$(\frac{\partial\Psi}{\partial n})_{n=\infty}=\frac{5-3i}{4i\omega}q_{s1}^{(\infty)}\frac{\partial}{\partial s}q_{s1}^{(\infty)*} \quad (10)$$

where $q_{s1}^{(\infty)}$ denotes the first-order tangential velocity at outer edge of the boundary-layer at the fixed wall and superscript * denotes complex conjugate of the value.

The boundary condition at free-surface is given as follow, which is interpreted as zero-tangential stress at free-surface.

$$(\frac{\partial^2\Psi}{\partial n^2})_{n=\infty}=\frac{4}{i\omega}\frac{\partial q_{n1}^{(0)}}{\partial s}\frac{\partial q_{s1}^{(0)*}}{\partial s}. \quad (11)$$

2.2 A wave maker problem

Let us denote oscillations of a flap type wave maker at $x=0$ as shown in Fig.1:

$$\Theta(t)=Re\{\theta_0e^{i\omega t}\} \quad (12)$$

The velocity potential $\Phi(x,z,t)$ is expressed as:

$$\Phi(x,z,t)=Re\{i\omega\theta_0\phi_1(x,z)e^{i\omega t}\} \quad (13)$$

The first-order potential $\phi_1(x,z)$ is given by the eigen function expansions:

$$\begin{aligned}\phi_1(x,z) &= ia_0\frac{K}{k}\frac{\cosh k(z-h)}{\cosh kh}e^{-ikx}\\ &\quad+\sum_{n=1}^{\infty}a_n\frac{K}{k_n}\frac{\cos k(z-h)}{\cos k_nh}e^{-k_nx} \quad (14)\end{aligned}$$

Corresponding stream function is expressed by:

$$\begin{aligned}\psi_1 &= a_0\frac{K}{k}\frac{\sinh k(z-h)}{\sinh kh}e^{-ikx}\\ &\quad+\sum_{n=1}^{\infty}a_n\frac{K}{k_n}\frac{\sin k(z-h)}{\sin k_nh}e^{-k_nx} \quad (15)\end{aligned}$$

where,

$$\begin{aligned}K=\omega^2/g &= k\tanh kh \quad (16)\\ &= -k_n\tan k_nh \quad (n=1,2,\cdots) \quad (17)\end{aligned}$$

The free-surface elevation is given by:

$$\eta(x)=-K\theta_0\{ia_0e^{-ikx}+\sum_{n=1}^{\infty}a_ne^{-k_nx}\} \quad (18)$$

Unknowns $a_j,(j=0,1,2,\cdots)$ are determined by the boundary condition at $x=0$ as:

$$\left.\begin{aligned}\frac{\partial}{\partial x}\phi_1(0,z) &= d-z, \quad 0\le z\le d\\ &= 0, \quad d<z\le h\end{aligned}\right\} \quad (19)$$

Let the stream function split into the progressing wave and local waves as follows:

$$\psi_1(x,z)=\psi_0(x,z)+\sum_{n=1}^{\infty}\psi_{Ln}(x,z) \quad (20)$$

The governing equation of a stream function for the mass transport in the exterior of the boundary-layers is given by eq.(9) and the boundary conditions on the free-surface, water bottom and the flap surface are summarized as:

$$\begin{aligned}\frac{\partial^2}{\partial z^2}\Psi(x,0) &= Re\{\frac{4}{i\omega}\frac{\partial w}{\partial x}\frac{\partial u^*}{\partial x}\} \quad (21)\\ \frac{\partial}{\partial z}\Psi(x,h) &= Re\{\frac{5-3i}{4i\omega}u\frac{\partial u^*}{\partial x}\} \quad (22)\\ \frac{\partial}{\partial x}\Psi(0,z) &= -Re\{\frac{5-3i}{4i\omega}w\frac{\partial w^*}{\partial z}\} \quad (23)\end{aligned}$$

where, $\nabla\psi_1=(-w,u)$

The total horizontal flow due to the mass transport must be zero, so that

$$\Psi(x,0)=\Psi(x,h)=\Psi(0,z)=0 \quad (24)$$

Furthermore, Ψ must be bounded at $x\to+\infty$.

Next, we split $\Psi(x,z)$ into four parts as:

$$\begin{aligned}\Psi &= \Psi_1(\psi_0,\psi_0)+\sum_n\Psi_{2n}(\psi_0,\psi_{Ln})\\ &\quad+\sum_n\sum_m\Psi_{3nm}(\psi_{Ln},\psi_{Lm})+\Psi_4(x,z) \quad (25)\end{aligned}$$

Solutions for Ψ_1, Ψ_{2n} and Ψ_{3nm} are obtained analytically if we allow them free from boundary condition imposed at $x = 0$. Ψ_1 is already given by Longuet-Higgins(1953).

Ψ_{2n} denotes interactions between ψ_0 and ψ_{Ln} and is expressed by:

$$\Psi_{2n} = \frac{a_0 a_n K^2}{2\omega} \cdot \frac{e^{-k_n x}}{\sinh kh \sin k_n h} \Big[\sin kx \times \times \{\cosh k(z-h) \sin k_n(z-h) + Z_s(z)\} + + \cos kx \{\sinh k(z-h) \cos k_n(z-h) + + Z_c(z)\} \Big] \qquad (26)$$

Substituting eq.(26) into eq.(9), we obtain two equations with respect to Z_s and Z_c which yield an eigenvalue equation. Solving the eigenvalue equation, we obtain

$$\left. \begin{array}{c} Z_s(z) \\ Z_c(z) \end{array} \right\} = \sum_{j=1}^{4} (A_j + B_j z) e^{\lambda_j z} \qquad (27)$$

where

$$\lambda_j = \pm k \pm i k_n, \qquad (j=1 \sim 4) \qquad (28)$$

Unknowns A_j, B_j are determined by boundary conditions at free-surface and water bottom.

Ψ_{3nm} denotes interactions between ψ_{Ln} and ψ_{Lm}. Assuming a solution of the form as:

$$\Psi_{3nm} = Z_L(z) e^{-(k_n+k_m)x} \qquad (29)$$

Substituting eq.(29) into eq.(9), we obtain

$$Z_L(z) = (C_1 + C_2 z) \cos k_{nm} z + (C_3 + C_4 z) \sin k_{nm} z \quad (30)$$

where $k_{nm} = k_n + k_m$.
Unknowns C_j (j=1 $\sim$ 4) are determined by boundary conditions at free-surface and water bottom.

Ψ_4 is obtained so that Ψ satisfies the boundary conditions at $x = 0$, and satisfies homogeneous condition at other boundaries. Assume a solution of the form for Ψ_4

$$\Psi_4(x, z) = \sum_{j=1}^{N} \{D_j(z-h) + E_j x\} e^{-\alpha_j x} \sin \alpha_j (z-h) \qquad (31)$$

where $\alpha_j = j\pi/h$
Then unknowns D_j and E_j, can be determined from known values of Ψ_1, Ψ_2 and Ψ_3 at $x = 0$.

Finally, mass transport velocity is obtained by eq.(2).

3 Experiments

Experiments were conducted at a narrow wave tank($L \times B \times D = 25m \times 0.6m \times 1.0m$) with a flap type wave maker. Water depth set $0.3m$. Fluid velocity were measured by an electro-magnetic velocity meter at several points in regular waves. Eulerian currents were obtained by averaging of measured records for 70 seconds. Wave height was about $0.05m$ and Eulerian currents were less than $0.03m/s$ in the experiments.

4 Results and discussion

Fig.2 shows calculated stream lines of mass transport, where four components and total stream lines are presented. Solid lines denote positive stream lines which circulate clockwise and dotted lines do counterclockwise. Ψ_1 is most dominant just away from the wave maker. Ψ_2 and Ψ_4 induce strong circulation near the wave maker, but they are opposite direction so that they weaken each other. Effect by Ψ_3 is small in this case, but they may depend on the wave number.

Fig.3 shows a comparison of Eulerian velocity distribution between measurements and calculation. Plotted vectors are normalized by

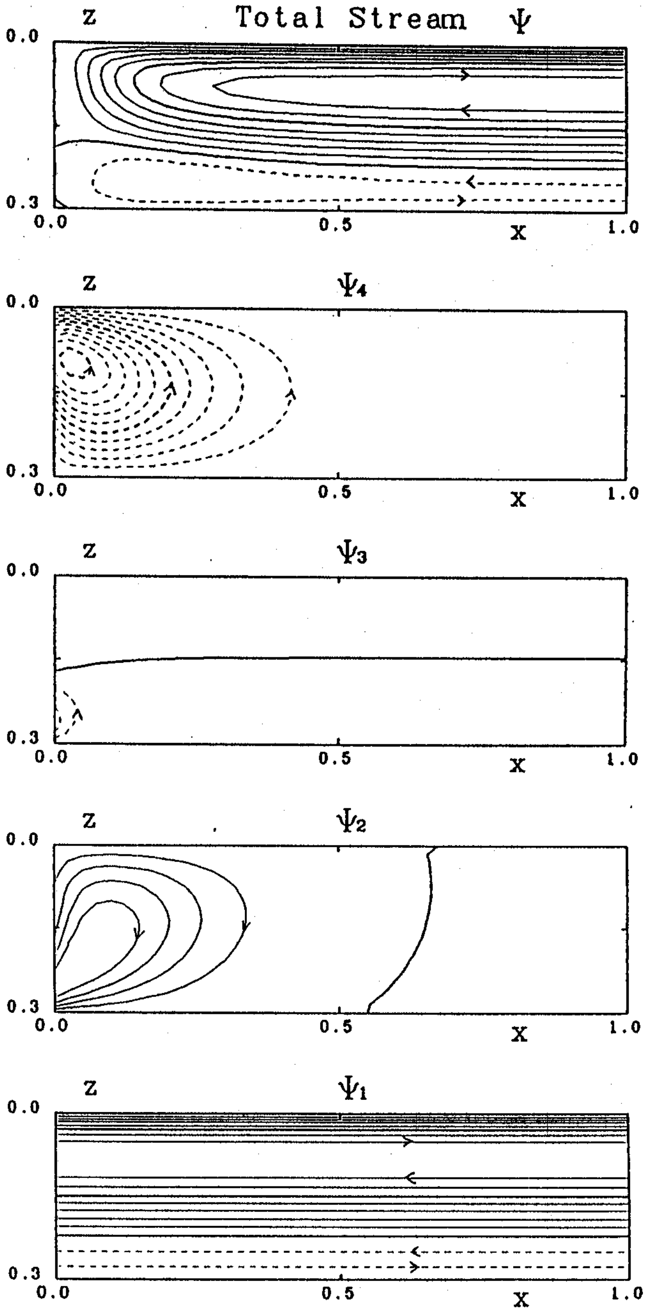

Fig.2 Calculated stream lines of mass transport ($T_W = 1.0sec$)

$$\overline{u} = \frac{\overline{U}}{\left(\frac{a_0^2 K^2 \omega k}{4 \sinh^2 kh}\right)} \qquad (32)$$

Measurements may contain some contradictions due to measuring accuracy, but general tendencies seem to agree with calculations.

Mass transport may be estimated by Eulerian measurements plus Stokes drift calculated from the theory, because the first-order quantities are known to agree with the theory in this problem. Fig.4 shows a comparison of mass transport velocities thus obtained. There exists up-stream in front of a wave maker, which agrees with observations of dye motion. At some distance from a wave maker, local disturbances disappear rapidly.

There are two currents near bottom and free-surface which flow as same direction as waves. Bottom current could not be explaind by the potential approach.(Hudspeth and Sulisz(1991))

References

[1] Longuet-Higgins,M.S.(1953): Philo. Trans. Royal Soc. London, Series A, Vol.245, No.903,pp.535-581.

[2] Iskandarani,M. and Liu,P.L.F.(1991): J. Fluid Mech., Vol.231, pp.395-415.

[3] Hudspeth,R.T. and Sulisz,W.(1991): J. Fluid Mech., Vol.230, pp.209-229.

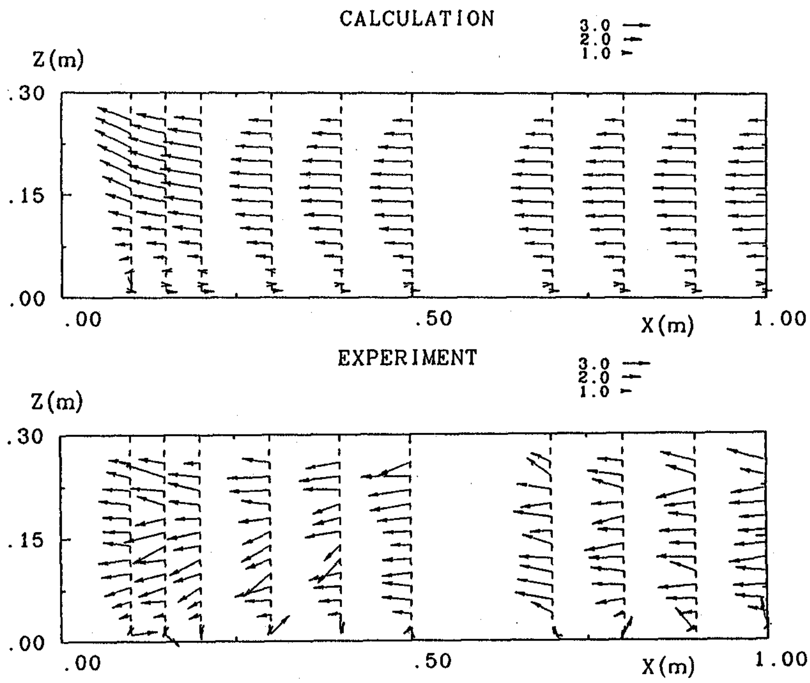

Fig.3 Vectors of Eulerian current ($T_W = 1.0sec$)

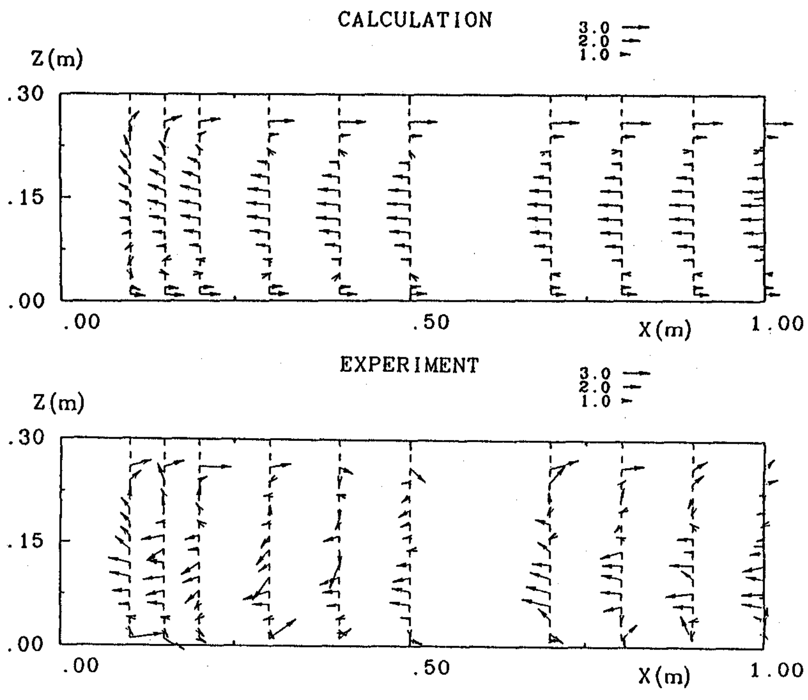

Fig.4 Vectors of mass transport velocity ($T_W = 1.0sec$)

DISCUSSION

Roberts A.J.: I wonder how you computed estimates of Lagrangian velocities from the Eulerian measurements of velocity in the experiments, this seems to me to be a difficult process.

Kyozuka Y.: Stokes drift assumed to be calculated by the amplitude of the propagating waves, because, according to my knowledge, the first-order quantities are known to agree well with theory in a wave maker problem. Then, mass trasnport is obtained by adding Stokes drift to Eulerian measurements, because my main concerns are on mass transport by waves.

Tulin M.: I want to congratulate Prof. Kyozuka for studying these wave induced turbulent circulatory flows in the wave tank. They are usually ignored, although we do not actually know their effect on the waves. Have you considered pumping the drift flow from the down-stream end, back to the wave maker?

Kyozuka Y.: The drift flow from the down-stream end is possible if there exist the reflected waves from the beach. They will produce partial circulations due to partial standing waves. As the reflected wave coefficients were less than a few parcent in our experiments, we think those effects might be negligible. We generated waves for 90 seconds, and analyzed the data for 70 seconds in the middle of measurements.

Choi H.S.: You said in your talk that you obtained the Stokes solution for the biharmonic in the boundary layer (laminar) near the bottom. The Stokes solution in two dimensions contains a logarithmic term, which causes trouble for the fluid region outside the boundary layer. Would you tell me why you treated this term?

Kyozuka Y.: In the first-order problem in the interior of the boundary layer, N-S equation is witten by

$$(\frac{\partial}{\partial t} - \nu\frac{\partial^2}{\partial n^2})\frac{\partial \psi_1}{\partial n} = \frac{\partial}{\partial t} q_{S1}^{(\infty)}$$

Since the first-order motion is simple harmonic, we have a following solution which satisfies the boudary conditions at $n = 0$ and $n = \infty$

$$\frac{\partial \psi_1}{\partial n} = q_{S1}^{(\infty)}(1 - e^{-\alpha n})$$

where

$$\alpha = (i\omega/\nu)^{1/2}$$

In our analysis, we didn't include a singular part of Stokes solution because there is no reason to include a singular part in this problem.