A Continuation Method for Computing Non-linear 3-D Free Surface Flows*

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1 Introduction

The subject of this paper is a pseudo-arclength continuation method for computing non-linear three-dimensional steady potential flow around a submerged body moving in a infinitely deep liquid at constant speed and distance below the free surface. Most numerical techniques for this problem are based on direct application of Newton's method to find the non-linear solution, [3]. Usually, these iterative methods use the solution of the double-body problem or the Kelvin–Helmholtz problem as initial guess. However, it is well known that Newton's method will only converge if the initial guess is sufficiently close to the solution of the non-linear problem. Hence, it is not surprising that severe convergence problems have been accounted with these methods.

In the present approach we instead embed the full non-linear problem in a sequence of problems such that one value of the embedding parameter corresponds to the Kelvin–Helmholtz problem, and another yields the full non-linear problem. The solution of the full non-linear problem is approached by gradually changing the value of the embedding parameter, and using the solution at the previous value as initial guess in the iterative Newton procedure. This is a well known technique for solving general non-linear problems [2] and it has for example been used to compute two-dimensional periodic water waves [1], [4]. However, it has apparently not been applied to the present problem before.

To describe the method in more detail, we introduce the following notation. Let the speed of the body be U, the acceleration of gravity be g, the velocity potential be ϕ and the elevation of the free surface be η. We describe the motion in Cartesian coordinates which are fixed with respect to the body. The x-axis points opposite to the forward direction, the z-axis is directed vertically upwards, and the y-axis is directed sidewise such that (x, y, z) forms a right-handed coordinate system. We embed the solution in a parameter, $0 \leq \alpha \leq 1$, such that $\alpha = 0$ yields the Kelvin-Helmholtz problem and $\alpha = 1$ corresponds to the full non-linear problem.

The velocity potential is governed by $\nabla^2\phi = 0$ in the interior and is subject to the following boundary conditions:

$$(1-\alpha)U(\phi_x - U) + \alpha\left(|\nabla\phi|^2 - U^2\right)/2 + g\eta = 0, \quad (1)$$

$$(1-\alpha)U\eta_x + \alpha(\phi_x\eta_x + \phi_y\eta_y) - \phi_z = 0, \quad (2)$$

on $z = \alpha\eta(x, y)$. On the body, we impose $\phi_n = 0$, and at infinity we require the potential to satisfy $(\phi_x, \phi_y, \phi_z) = (U, 0, 0)$. We also require the radiation condition to be fulfilled, i.e. no waves should be present ahead of the body.

The potential in the interior is uniquely determined by its values on the boundary. It is therefore sufficient to consider the above equations as a problem for the boundary values of ϕ together with the surface elevation. Furthermore, the surface elevation can be eliminated between the boundary conditions

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on the free surface. Therefore, the solution is completely characterized by the values of ϕ on the boundary. Below, we will use the abstract notation $L(\phi,\alpha) = 0$ for this non-linear partial differential equation together with the boundary conditions.

2 Pseudo-arclength continuation

Following [2], the solution will be considered as a function of the pseudo-arclength s, $\phi = \phi(s)$ and $\alpha = \alpha(s)$. Assume that a solution point (ϕ_0, α_0) is known, and let it have pseudo-arclength s_0. We define the pseudo-arclength relative to that point by

$$s = s_0 + \langle \dot\phi_0, \phi - \phi_0 \rangle + \dot\alpha_0(\alpha - \alpha_0), \tag{3}$$

where $\langle \cdot, \cdot \rangle$ is the L_2 scalar product. The tangent $(\dot\phi_0, \dot\alpha_0)$ is the solution of

$$L_\phi[\phi_0, \alpha_0]\dot\phi_0 = -L_\alpha[\phi_0, \alpha_0]\dot\alpha_0, \tag{4}$$

subject to the normalization $\langle \dot\phi_0, \dot\phi_0 \rangle + \dot\alpha_0^2 = 1$. The direction of the tangent is determined by requiring the scalar product between the previous and the present tangent to be positive.

We augment $L[\phi, \alpha] = 0$ by the arclength equation $N[\phi, \alpha; s] = 0$, where

$$N[\phi, \alpha; s] = \langle \dot\phi_0, \phi - \phi_0 \rangle + \dot\alpha_0(\alpha - \alpha_0) - (s - s_0). \tag{5}$$

We use the predictor $(\phi^0, \alpha^0) = (\phi_0 + \dot\phi_0 \Delta s, \alpha_0 + \dot\alpha_0 \Delta s)$ as initial guess for the solution at $s = s_0 + \Delta s$. The predictor is corrected by Newton's method on the augmented system, where the improvements of the solution are found by solving

$$\begin{pmatrix} L_\phi[\phi^k, \alpha^k] & L_\alpha[\phi^k, \alpha^k] \\ N_\phi[\phi^k, \alpha^k] & N_\alpha[\phi^k, \alpha^k] \end{pmatrix} \begin{pmatrix} \Delta\phi^k \\ \Delta\alpha^k \end{pmatrix} = - \begin{pmatrix} L[\phi^k, \alpha^k] \\ N[\phi^k, \alpha^k] \end{pmatrix} \tag{6}$$

The solution is then updated according to

$$\phi^{k+1} = \phi^k + \Delta\phi^k, \tag{7}$$

$$\alpha^{k+1} = \alpha^k + \Delta\alpha^k. \tag{8}$$

The iteration is truncated when $||\phi^{k+1} - \phi^k|| + |\alpha^{k+1} - \alpha^k| < \epsilon$.

If the iteration converges, we repeat the procedure until $\alpha = 1$ is reached. The number of iterations that was required to get convergence is used to determine next step-size Δs. However, if the iteration diverges, we halve the step-size and try again. By the implicit function theorem, the iteration must converge for a sufficiently small step-size if the Jacobian $L_\phi[\phi_0, \alpha_0]$ is non-singular. A singular Jacobian corresponds to a turning point or a bifurcation point. The method can be extended to handle these cases as well, but we refer to [2] for details.

3 The numerical method

We represent the velocity potential in terms of a single layer distribution,

$$\phi(P) = \int_S \frac{\sigma(Q)}{|P - Q|} \, dS(Q) + Ux. \tag{9}$$

Here, $P = (x, y, z)$, $Q = (\tilde x, \tilde y, \tilde z)$ and $|P - Q|$ is the distance between P and Q. This integral is discretized by first truncating the infinite domain to a finite domain and then applying a panel method where σ is approximated to vary linearly over each panel. We get the discrete counterpart of $L(\phi, \alpha)$ by enforcing the boundary conditions at the free surface and at the body to be satisfied at a finite number of control points.

The work described here is still in progress and we have not yet completed the implementation of the method. Therefore, no numerical examples are presently available. However, we anticipate to be able to show non-trivial numerical examples during the workshop.

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