Boundary integral equations for bodies of small, but finite, thickness

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1 Introduction

The reduction of linear wave-body interaction problems to integral equations over the surface of the body is a commonplace. However, it is also well known that the standard integral equations are degenerate when the body is thin (see, e.g. Warham (1988) or Martinez (1991)). This degeneration is discussed and two methods for its elimination are described.

2 Formulation

We start with a fundamental solution (in three dimensions),

$$G(P,Q) = -1/(2\pi R) + G_1(P,Q)$$

where R is the distance between the two points P and Q. G_1 is harmonic and bounded in the water, and is chosen so that G satisfies the free-surface and radiation conditions; the simplest choice for G_1 is

$$G_1(P,Q) = G_1(x,y,z;\xi,\eta,\zeta) = \frac{-1}{2\pi}\int_0^\infty \frac{k+K}{k-K} J_0(k\rho)\, e^{-k(y+\eta)}\, dk,$$

where $y = 0$ is the free surface and y points down, $K = \omega^2/g$, $\rho^2 = (x-\xi)^2 + (z-\zeta)^2$ and the integration contour is indented below the pole of the integrand at $k = K$. Then, for a direct method, one uses Green's theorem to obtain various boundary integral equations for the boundary values of the potential ϕ.

To fix ideas, let us consider a scattering problem for a (fixed) body with wetted surface S. Let ϕ^{in} be the incident potential, let ϕ^{sc} be the scattered potential, and let

$$\phi = \phi^{\text{in}} + \phi^{\text{sc}}$$

be the total potential, satisfying

$$\frac{\partial\phi}{\partial n} = 0 \qquad \text{on } S,$$

where we take the unit normal on S, $\boldsymbol{n}$, pointing into the water. Then, we have

$$2\phi^{\text{sc}}(P) = -\int_S \phi(q)\frac{\partial}{\partial n_q} G(P,q)\, ds_q, \tag{2.1}$$

giving the scattered field at any point P in the water in terms of the boundary values of the total potential, $\phi(q)$. Letting P go to p on S gives the familiar integral equation,

$$\phi(p) + \int_S \phi(q)\frac{\partial}{\partial n_q} G(p,q)\, ds_q = 2\phi^{\text{in}}(p).$$

This is a Fredholm integral equation of the second kind with a weakly-singular kernel; we write it concisely as

$$(I + K)\phi = 2\phi^{\text{in}}. \tag{2.2}$$

3 Burton and Miller

If the body pierces the free surface, the integral equation (2.2) will suffer from *irregular frequencies.* One way to eliminate these is to adapt the method used by Burton & Miller (1971) in acoustics. Thus, evaluate the normal derivative of ϕ^{sc} on S, using (2.1), to give

$$\frac{\partial}{\partial n_p}\int_S \phi(q)\frac{\partial}{\partial n_q}G(p,q)\,ds_q = 2v^{\text{in}}(p).$$

where $v^{\text{in}} = \partial\phi^{\text{in}}/\partial n$; write this hypersingular integral equation as

$$N\phi = 2v^{\text{in}}. \tag{3.1}$$

Then, Burton and Miller's method consists of solving a linear combination of (2.2) and (3.1),

$$(I + K + \alpha N)\phi = 2(\phi^{\text{in}} + \alpha v^{\text{in}}), \tag{3.2}$$

where the coupling parameter α is usually taken as $\alpha = \mathrm{i}\gamma$, with γ real and non-zero. For water waves, this method has been investigated by Lee & Sclavounos (1989). We shall return to (3.2), the *BM equation*, later.

4 Thin bodies

Consider a submerged pancake of (small) nominal thickness h. Let S_+ (S_-) be the 'upper' ('lower') piece of S, so that $S = S_+ \cup S_-$, and write $p_\pm$ and $q_\pm$ for points on $S_\pm$ (see Figure 1). Define

$$K_+\phi_+ = \int_{S_+} \phi(q_+)\frac{\partial}{\partial n_q^+}G(p_+,q_+)\,ds_q^+ \quad \text{and} \quad K_-\phi_- = \int_{S_-} \phi(q_-)\frac{\partial}{\partial n_q^-}G(p_-,q_-)\,ds_q^-,$$

where $\phi_\pm = \phi(q_\pm)$, $ds_q^\pm$ is the surface element at $q_\pm$ and $\partial/\partial n_q^\pm$ denotes normal differentiation at $q^\pm$ into the water. In the integral operators $K_\pm$, both the field point $p_\pm$ and the source point $q_\pm$ are on the *same* surface, namely $S_\pm$. We also need similar operators in which the field point and source point are on *different* surfaces: define

$$K_{-+}^h\phi_+ = \int_{S_+} \phi(q_+)\frac{\partial}{\partial n_q^+}G(p_-,q_+)\,ds_q^+ \quad \text{and} \quad K_{+-}^h\phi_- = \int_{S_-} \phi(q_-)\frac{\partial}{\partial n_q^-}G(p_+,q_-)\,ds_q^-,$$

where the superscript h reminds us that the two surfaces are separated by a distance h. With all this notation, we can rewrite the integral equation (2.2) as

$$\left.\begin{aligned} (I + K_+)\phi_+ + K_{+-}^h\phi_- &= 2\phi_+^{\text{in}} \quad \text{on } S_+ \\ K_{-+}^h\phi_+ + (I + K_-)\phi_- &= 2\phi_-^{\text{in}} \quad \text{on } S_-. \end{aligned}\right\} \tag{4.1}$$

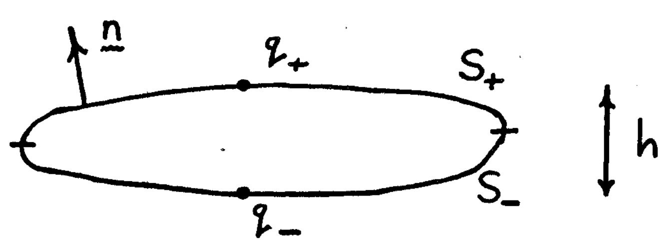

Figure 1: The submerged pancake, with surface partitioned as $S = S_+ \cup S_-$.

Now, what happens as $h \to 0$, so that S_+ and S_- collapse on to the same surface, S_+ say? Clearly,

$$K_-\phi_- \to -K_+\phi_-$$

since $\boldsymbol{n}(q_+) = -\boldsymbol{n}(q_-)$. Also, using the jump relations, we find that

$$K^h_{-+}\phi_+ \to \phi_+ + K_+\phi_+ \qquad \text{and} \qquad K^h_{+-}\phi_- \to \phi_- + K_-\phi_- = \phi_- - K_+\phi_-.$$

So, in this limit, (4.1) becomes

$$\left.\begin{aligned} (I + K_+)\phi_+ + (I - K_+)\phi_- &= 2\phi^{\text{in}}_+ \quad \text{on } S_+ \\ (I + K_+)\phi_+ + (I - K_+)\phi_- &= 2\phi^{\text{in}}_- \quad \text{on } S_-. \end{aligned}\right\} \qquad (4.2)$$

However, ϕ^{in} is blissfully unaware of the pancake's presence, so we have $\phi^{\text{in}}_+ = \phi^{\text{in}}_-$ in the limit, whence the pair of equations (4.2) degenerates into just one equation, namely

$$\langle\phi\rangle + K_+[\phi] = 2\phi^{\text{in}}, \qquad (4.3)$$

where $\langle\phi\rangle = \phi_+ + \phi_-$ and $[\phi] = \phi_+ - \phi_-$. Thus, we obtain one equation for two unknowns.

We can make a similar analysis for the hypersingular equation (3.1), rewriting it as the pair

$$\left.\begin{aligned} N_+\phi_+ + N^h_{+-}\phi_- &= 2v^{\text{in}}_+ \quad \text{on } S_+ \\ N^h_{-+}\phi_+ + N_-\phi_- &= 2v^{\text{in}}_- \quad \text{on } S_-, \end{aligned}\right\} \qquad (4.4)$$

where

$$N_+\phi_+ = \frac{\partial}{\partial n^+_p}\int_{S_+}\phi(q_+)\frac{\partial}{\partial n^+_q}G(p_+,q_+)\,ds^+_q, \quad N_-\phi_- = \frac{\partial}{\partial n^-_p}\int_{S_-}\phi(q_-)\frac{\partial}{\partial n^-_q}G(p_-,q_-)\,ds^-_q,$$

$$N^h_{-+}\phi_+ = \frac{\partial}{\partial n^-_p}\int_{S_+}\phi(q_+)\frac{\partial}{\partial n^+_q}G(p_-,q_+)\,ds^+_q, \quad N^h_{+-}\phi_- = \frac{\partial}{\partial n^+_p}\int_{S_-}\phi(q_-)\frac{\partial}{\partial n^-_q}G(p_+,q_-)\,ds^-_q.$$

As $h \to 0$, we find that $N_- \to N_+$, $N^h_{-+} \to -N_+$ and $N^h_{+-} \to -N_+$; since $v^{\text{in}}_+ = -v^{\text{in}}_-$ in this limit, we find that the pair (4.4) degenerates into a single equation for $[\phi]$, namely

$$N_+[\phi] = 2v^{\text{in}}_+. \qquad (4.5)$$

This is precisely the hypersingular integral equation for thin plates, discussed at previous Workshops; see Parsons & Martin (1992).

5 Non-degenerate equations for thin bodies

We describe two non-degenerate methods.

5.1 Mixed system

Solve the pair of equations obtained by combining the first of (4.1) with the second of (4.4),

$$\left.\begin{aligned} (I + K_+)\phi_+ + K^h_{+-}\phi_- &= 2\phi^{\text{in}}_+ \quad \text{on } S_+ \\ N^h_{-+}\phi_+ + N_-\phi_- &= 2v^{\text{in}}_- \quad \text{on } S_-. \end{aligned}\right\} \qquad (5.1)$$

This pair is uniquely solvable for all $h \geq 0$; as $h \to 0$, it reduces directly to (4.3) and (4.5). If the body pierces the free surface, (5.1) has irregular frequencies (which are identifiable as the eigenvalues of a certain interior problem). Numerical experiments with the pair (5.1) in acoustics have been described by Krishnasamy *et al.* (1993). We remark that (5.1) can be viewed as a pair of *dual integral equations* (see, for example, Sneddon (1966)).

5.2 Burton and Miller

Consider the BM equation, (3.2). Partitioning S as before gives

$$\left.\begin{aligned} (I + K_+ + \alpha N_+)\phi_+ + (K^h_{+-} + \alpha N^h_{+-})\phi_- &= 2(\phi^{\text{in}}_+ + \alpha v^{\text{in}}_+) \quad \text{on } S_+ \\ (K^h_{-+} + \alpha N^h_{-+})\phi_+ + (I + K_- + \alpha N_-)\phi_- &= 2(\phi^{\text{in}}_- + \alpha v^{\text{in}}_-) \quad \text{on } S_-. \end{aligned}\right\} \tag{5.2}$$

Letting $h \to 0$, as before, these reduce to

$$\left.\begin{aligned} (I + K_+ + \alpha N_+)\phi_+ + (I - K_+ - \alpha N_+)\phi_- &= 2(\phi^{\text{in}}_+ + \alpha v^{\text{in}}_+) \quad \text{on } S_+ \\ (I + K_+ - \alpha N_+)\phi_+ + (I - K_+ + \alpha N_+)\phi_- &= 2(\phi^{\text{in}}_+ - \alpha v^{\text{in}}_+) \quad \text{on } S_-. \end{aligned}\right\} \tag{5.3}$$

The sum of these two equations gives twice (4.3) whereas the difference gives (4.5) multiplied by 2α. Hence, the BM equation is not degenerate for thin bodies provided that $\alpha \neq 0$.

We observe that solving (3.2) requires about twice as much work as solving the mixed system (5.1). On the other hand, (3.2) does not have any irregular frequencies when the body is floating and it does not require a partitioning of the wetted surface.

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