The Statistics of Irregular Second-order Waves

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An important factor in designing offshore structures is to determine its required deck elevation. Generally, the requirement will be that the expected largest waves in the design sea state, or alternatively a deterministic design wave, shall not hit the deck. The deterministic wave approach is easier to use and by applying a nonlinear wave theory like for example a higher order Stokes theory, the wave skewness is accounted for. It seems to be generally recognized, however, that crest heights are overpredicted by this approach, thus resulting in conservative deck elevation requirements.

The present study deals with the skewness of irregular second-order waves in finite water depth. It is shown that the slowly-varying part of the second-order waves, not present in a deterministic approach, gives a negative contribution to the wave skewness. This complies with the observation by Longuet-Higgins and Stewart [4], that the slowly varying sea level is depressed in a group of large waves. A physical explanation in terms of radiation stresses is given in the same reference. The effect becomes stronger as the water depth decreases and thus reduces the increased skewness often assumed for shallower waters. Theoretical predictions are compared with field measurements in the Ekofisk area in the North Sea. Differences between buoy and radar measurements are discussed, and a different second order elevation is used when comparing the theory with buoy measurements, accounting for the fact that the buoy estimates the surface elevation based on measured accelerations.

An irregular sea is usually written as a Fourier sum

$$\Phi^{(1)} = \sum_{j=-N}^{N} \frac{iga_j}{2\omega_j} \frac{\cosh k(z+h)}{\cosh kh} e^{i\psi_j} \tag{1}$$

$$\eta^{(1)} = \sum_{j=-N}^{N} \frac{a_j}{2} e^{i\psi_j} \tag{2}$$

where the amplitudes are related to the (one-sided) wave spectrum $S_\eta(\omega)$

$$a_j = \sqrt{2S_\eta(|\omega_j|)\Delta\omega} \tag{3}$$

and the phase functions are

$$\psi_j = \omega_j t - k_j x - \epsilon_j \tag{4}$$

ϵ_j are random phase angles, uniformly distributed on the interval $[0,2\pi)$. Real quantities are assured by requiring $\omega_{-j} = -\omega_j$ and $\epsilon_{-j} = -\epsilon_j$. Typically we shall have $\omega_j = (2j - \text{sgn} j)\Delta\omega/2$.

The second order quantities depend on terms that are products of sums like (1) and (2), and we can write

$$\Phi^{(2)} = \sum_{i,j=-N}^{N} \frac{ia_i a_j}{2} P_{ij} \frac{\cosh(k_i+k_j)(z+h)}{\cosh(k_i+k_j)h} e^{i(\psi_i+\psi_j)} + \sum_{j=1}^{N} \frac{a_j^2 g k_j t}{2\sinh 2k_j h} t \tag{5}$$

$$\eta^{(2)} = \sum_{i,j=-N}^{N} \frac{a_i a_j}{2} E_{ij} e^{i(\psi_i+\psi_j)} \tag{6}$$

with the following expressions for P_{ij} and E_{ij}

$$P_{ij} = (1-\delta_{-i,j}) \frac{\frac{g^2 k_i k_j}{2\omega_i\omega_j} - \frac{1}{4}(\omega_i^2 + \omega_j^2 + \omega_i\omega_j) + \frac{g^2}{4}\frac{\omega_j k_i^2 + \omega_i k_j^2}{\omega_i\omega_j(\omega_i+\omega_j)}}{(\omega_i+\omega_j) - g\frac{k_i+k_j}{\omega_i+\omega_j}\tanh(k_i+k_j)h} \tag{7}$$

$$E_{ij} = \frac{1}{g}(\omega_i+\omega_j)P_{ij} - (1-\delta_{-i,j})\left[\frac{gk_ik_j}{4\omega_i\omega_j} - \frac{1}{4g}(\omega_i^2+\omega_j^2+\omega_i\omega_j)\right] \tag{8}$$

The Kroenecker delta ($\delta_{-i,j} = 1$ if $i+j=0$, zero otherwise) is introduced to avoid a singular P_{ij} and an undesired additive constant in $\eta^{(2)}$. In deriving the expressions for P_{ij} and E_{ij} only the symmetric parts have been retained, since the skew-symmetric parts would vanish in the summation.

Eq. (7) and (8) express $\Phi^{(2)}$ and $\eta^{(2)}$ in terms of quadratic transfer-functions P_{ij} and E_{ij}. We can therefore study the statistics of the surface elevation using the method developed by Kac and Siegert [2], for second order Volterra series. This method was introduced to hydrodynamics by Neal [5], in the study of second order oscillations of floating structures. In principle the full probability distribution for the combined first and second order process can be obtained by this method, but to remain consistent with the perturbation expansion to second order, moments of the distribution higher than third order should be discarded.

A study of the statistics of nonlinear deep water waves by the method of Gram-Charlier series is found in a classical paper by Longuet-Higgins [3]. In particular the skewness is studied in some detail and it is shown that the skewness of second order deep water waves is in general positive. It is possible to extend that proof to show that the sum-frequency terms will give a positive contribution to the skewness while the difference-frequency terms will reduce it.

The Volterra series method has been used to obtain the skewness values for various sea states and compared to measurements of wave elevations in the Ekofisk area of the North Sea. Both radar and buoy measurements are available, and to compare with the radar measurements the transfer function E_{ij} given by equation (8) has been used. For the buoy, however, som extra analysis is required. A wave rider buoy measures accelerations and calculates the wave elevation by some algorithm, most likely based on linear wave theory. This implies a second order elevation different from the "true" value given in (8).

Let the vertical acceleration be given by

$$a_z = \sum_{j=-N}^{N} \frac{a_j g_j}{2} e^{i\psi_j} + \sum_{i,j=-N}^{N} \frac{a_i a_j}{2} G_{ij} e^{i(\psi_i+\psi_j)} \tag{9}$$

This is what the buoy actually measures. If the buoy assumes linear linear wave theory, it will estimate a surface elevation

$$\eta_B = \sum_{j=-N}^{N} \frac{a_j g_j}{-2\omega_j^2} e^{i\psi_j} + \sum_{i,j=-N}^{N} \frac{a_i a_j}{2} \frac{G_{ij}}{-(\omega_i+\omega_j)^2} e^{i(\psi_i+\psi_j)} \tag{10}$$

The first order transfer function for the acceleration is simply $g_j = -1$, while the following expression is found for for G_{ij}

$$G_{ij} = -P_{ij}(k_i+k_j)\tanh(k_i+k_j)h - \frac{g}{4}(k_i^2+k_j^2) + \frac{gk_ik_j}{4}\left(\frac{\omega_i}{\omega_j} + \frac{\omega_j}{\omega_i}\right) - \frac{g}{4}\left(\frac{\omega_i k_j^2}{\omega_j} + \frac{\omega_j k_i^2}{\omega_i}\right) \tag{11}$$

H_S (m)	T_Z (s)	Radar 1984		Radar 1984-85		Calculated	Calculated, sum freq. only
		$\alpha_{3\eta}$	$\sigma_{\alpha_{3\eta}}$	$\alpha_{3\eta}$	$\sigma_{\alpha_{3\eta}}$		
3-4	5-6	0.128	0.072	0.121	0.063	0.205	0.314
3-4	6-7	0.117	0.064	0.110	0.065	0.149	0.234
3-4	7-8	0.092	0.069	0.092	0.059	0.113	0.182
4-5	6-7	0.170	0.031	0.169	0.049	0.188	0.298
4-5	7-8	0.138	0.079	0.129	0.075	0.144	0.233
4-5	8-9	0.112		0.050	0.090	0.113	0.183
5-6	7-8	0.159	0.081	0.159	0.081	0.173	0.283
5-6	8-9	0.135	0.070	0.124	0.071	0.137	0.223
5-6	9-10	0.133	0.016	0.126	0.016	0.113	0.194
6-7	9-10	0.159	0.059	0.163	0.053	0.133	0.228
6-7	10-11	0.148	0.048	0.142	0.040	0.115	0.207
7-8	9-10	0.209	0.013	0.209	0.013	0.152	0.263
7-8	10-11	0.141	0.047	0.131	0.036	0.131	0.239
8-9	10-11					0.147	0.270
8-9	11-12	0.168	0.033	0.140	0.051	0.132	0.256

Table 1: *Radar measurements and calculated values of skewness $\alpha_{3\eta}$ for various sea states. $\sigma_{\alpha_{3\eta}}$ is the standard deviation of the measured values.*

Another concern with the buoy data is that the difference frequency contributions to the acceleration are very small and may in fact be filtered out as noise in the measurements. One should therefore consider a low frequency cut-off in the calculations when comparing with buoy data.

In table 1 are shown skewness values obtained from radar measurements during 1984 and for 1984-85 combined. The data have been grouped into scatter diagrams and mean values and standard deviations for each of the groups are shown. Also shown are the calculated values. The rightmost column gives the calculated values when the difference-frequency terms are omitted. A JONSWAP spectrum with γ equal to 3 was used for all sea states. Varying γ within reasonable limits modifies the calculated skewnesses by 5-10%.

The agreement detween observed and calulated values of the skewness is surprisingly good. In all cases the calculated value lies within one standard deviation form the mean observed value. The table clearly shows that if the difference-frequency terms are not included, the skewness is overestimated, in this water depth roughly by a factor 1.5-2.

A comparison of skewness values measured by wave staffs in the Gulf of Mexico with theoretical predictions based on the theory of regular Stokes waves, thus excluding the difference-frequency terms, is reported by Arhan and Plaisted [1]. They actually find that their theory overpredicts the skewness by a factor 1.5-2.5. They attribute the difference to directional spreading of the waves.

Table 2 shows a similar comparison for the buoy data. The skewness is calculated both with a long period cutoff at 25 s, and with the difference frequency contributions excluded all together. More information about the buoy software is required to come up with more presise computed estimates.

It should be noted that other aspects of the performance of wave-rider buoys, like the tendency to cut the peaks of very steep waves and the buoy's motion transfer function, make buoy recordings unsuited for detailed second-order analysis. Another issue, the fact that the software used for converting the analog measured record to digital form may truncate the measured elevation, was studied and corrected

H_S (m)	T_Z (s)	Buoy 1984 $\alpha_{3\eta}$	Buoy 1984 $\sigma_{\alpha_{3\eta}}$	Calculated, 25 s cutoff	Calculated, sum freq. only
3-4	5-6	-0.007	0.059	-0.589	0.099
3-4	6-7	-0.029	0.060	-0.485	0.075
3-4	7-8	-0.108		-0.346	0.060
4-5	6-7	-0.025	0.053	-0.538	0.096
4-5	7-8	-0.008	0.067	-0.413	0.077
4-5	8-9			-0.220	0.061
5-6	7-8	-0.046	0.063	-0.465	0.094
5-6	8-9	0.003	0.088	-0.262	0.075
5-6	9-10	-0.163		-0.175	0.070
6-7	9-10	-0.057	0.073	-0.204	0.083
6-7	10-11			-0.134	0.082
7-8	9-10	0.202	0.082	-0.231	0.096
7-8	10-11	-0.032	0.091	-0.153	0.096
8-9	10-11	0.064	0.024	-0.172	0.108
8-9	11-12	-0.035	0.063	-0.102	0.112

Table 2: *Buoy measurements and calculated values of skewness* $\alpha_{3\eta}$ *for various sea states.* $\sigma_{\alpha_{3\eta}}$ *is the standard deviation of the measured values.*

for by Vinje [6]. This effect is most significant in smaller waves and is not accounted for in table 2.

It can also be noted that Arhan and Plaisted [1], present skewness values from bouy measurements that are significantly lower than those obtained from the wave staff measurements.

References

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