A STRONG VORTEX PLACED NEAR A FREE SURFACE

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We consider an inviscid fluid with a free surface. A line vortex with initial depth of submergence D is suddenly (impulsively) put into the fluid at time zero. The gravitational acceleration is g. The flow is two-dimensional. The vortex has circulation Γ, defined positive counter-clockwise. Let us examine the transient interaction between the vortex and the free surface.

Within inviscid theory, it is difficult to generate this single vortex. But it is possible, if we have an extremely slender foil to which the Kutta condition applies. At t=0 the foil starts impulsively to move downwards with a large velocity and a small angle of attack, so that it obtains a constant circulation $-\Gamma$. A starting vortex with circulation Γ is shed. Otherwise the influence from the foil on the free surface flow is negligible.

The fluid is at rest for $t < 0$. The vortex is assumed to be free, i.e. it moves with the fluid velocity according to Helmholtz's theorem. A cartesian coordinate system is defined, with x axis in the undisturbed free surface and y axis vertically upwards. We define $x = 0$ by the initial location of the vortex.

We introduce dimensionless quantities by defining D as unit of length, Γ/D as unit of velocity, and D^2/Γ as unit of time. We have one characteristic dimensionless number; the Froude number defined by:

$$F = \Gamma / (g D^3)^{1/2} \tag{1}$$

The dimensionless velocity potential is denoted by $\Phi(x,y,t)$. The vortex location is given by $x = X(t)$ and $y = -Y(t)$. The dimensionless surface elevation is denoted by $\eta(x,t)$. Our initial/boundary value problem, with a single free vortex, corresponds to that of Telste (1989) with two vortices. The solution is given in terms of a Taylor series in time. After the problem has been expanded in powers of t, we can write the total velocity potential as a sum,

$$\Phi = \psi + \phi \tag{2}$$

applying the principle of superposition to each order. Here ψ is the vortex potential, which satisfies $\psi = 0$ at $y = 0$ and takes care of the full singularity of the moving vortex point.

Its non-expanded version is then:

$$2\pi\psi = \arctan\frac{y+Y}{x-X} + \arctan\frac{y-Y}{x-X} \qquad (3)$$

In eq.(2) ϕ is the regular potential. It satisfies Laplace's equation for $y < 0$. The nonlinear terms are here represented by inhomogeneous conditions at $y = 0$, generated by lower order solutions. Taylor expansion in time defines the asymptotic series:

$$(\psi, \phi, \eta, X, Y) = (\psi_0, \phi_0, \eta_0, X_0, Y_0) \qquad (4)$$

$$+\, t\,(\psi_1, \phi_1, \eta_1, X_1, Y_1) + t^2\,(\psi_2, \phi_2, \eta_2, X_2, Y_2) + ..$$

By this approach we obtain the behaviour for later time by extrapolation from the initial instant $t = 0$, and all boundary conditions are applied at the undisturbed free surface $y = 0$.

The boundary value problems for the regular potential to each order are solved analytically by Poisson's integral formula for a half-plane. Exact integrations are carried out by residue calculus. By definition we have:

$$X_0 = 0\ ,\ Y_0 = 1\ ,\ \eta_0 = 0 \qquad (5)$$

The zeroth order vortex potential is then given by:

$$2\pi\psi_0 = \arctan\frac{y+1}{x} + \arctan\frac{y-1}{x} \qquad (6)$$

To the first order, the vortex moves in a straight line;

$$X_1 = 1/(4\pi)\ ,\ Y_1 = 0 \qquad (7)$$

with direction of motion opposite to that of a weak vortex, see Lamb (1932,p.223). The zeroth order regular potential is identically zero. The first order surface elevation is:

$$\eta_1 = \left(\frac{\partial\psi_0}{\partial y}\right)_{y=0} = \frac{x}{\pi\,(x^2+1)} \qquad (8)$$

The second order surface elevation is written as a sum of two contributions;

$$\eta_2 = \frac{1}{2}\left(\frac{\partial\psi_1}{\partial y} + \frac{\partial\phi_1}{\partial y}\right)_{y=0} = \frac{x^4 - 6x^2 + 1}{8\pi^2\,(x^2+1)^3} + \frac{x^2 - 1}{8\pi^2\,(x^2+1)^2}$$

$$= (4\pi^2)^{-1}\, x^2\,(x^2 - 3)\,/\,(x^2+1)^3 \qquad (9)$$

the first one from a static vortex and the second one from the actual free motion of the vortex. The second order position is:

$$X_2 = 0\ ,\ Y_2 = 0 \tag{10}$$

To second order the vortex moves in a straight line, as to first order. We note that the contributions to the second order position from the vortex potential as well as the regular potential are zero. The third order vortex position is given by:

$$X_3 = \frac{1}{24\pi}\left(\frac{1}{4\pi^2} - \frac{1}{F^2}\right)\ ,\ Y_3 = 0 \tag{11}$$

The critical vortex strength is defined by $X_3 = 0$:

$$F = F_c = 2\pi \tag{12}$$

If the Froude number is larger than this critical value, the vortex is supercritical (strong) and will accelerate in its initial direction of motion until breaking of the free surface occurs. For subcritical (weak) vortices with Froude number below the critical value, our theory shows that they will be decelerated at small times. A very weak vortex must turn and move steadily according to Lamb's solution when time tends to infinity. However, the present theory has a very restricted validity for weak vortices, and gives no information about such a transition to steady solution.

From the third order surface elevation, we only give the gravity dependent part. It has a sign difference compared with the first order elevation, because gravity waves extract potential energy:

$$\eta_3^{(gravity)} = - \frac{x}{3\pi^2 F^2 (x^2 + 1)^2} \tag{13}$$

The leading gravity wave is now defined as the asymptotic limit as $|x| \to \infty$. It dies out more quickly in space than the leading wave from a Cauchy-Poisson disturbance (Lamb 1932, p.385), which may be generated by an initial surface velocity, an initial surface elevation or an impulsively started source below the free surface.

When this source problem is solved by the present method (Tyvand 1990), it turns out that the leading nonlinear effect is to reduce the surface elevation in a region which has the width of 1.36 initial depths. If the source has negative strength (i.e. a sink), its surface trough will become deeper due to the same second-order effect. The leading gravitational effect on the surface elevation is of third order. This shows that a time scaling involving gravity will be incorrect for this transient problem. Thus the common Froude number expansion will not be useful for a strong impulsive source, cfr. Vanden-Broeck et al. (1978).

REFERENCES

Lamb, H. 1932 "Hydrodynamics". Cambridge Univ. Press.

Telste, J.G. 1989 Potential flow about two counter-rotating vortices approaching a free surface. J. Fluid Mech. 201, 259-278.

Tyvand, P.A. 1990 Transient free-surface flow due to a strong source. Manuscript.

Vanden-Broeck, J.M., Schwartz, L. & Tuck, E. 1978 Divergent low-Froude number series expansion of nonlinear free-surface flow problems. Proc. Roy. Soc. London A 361, 207-224.

DISCUSSION

Newman: Have you considered the linear time-domain solution, which could be developed starting from a vertical line of horizontal dipoles?

Tyvand: No. I have not done any linear theory in the time domain. This is of interest only for weak (subcritical) vortices. In the present work, I am concerned with the successive triggering of nonlinear effects shortly after an impulsive start. Two such levels of nonlinear interactions are needed to produce the exact analytical distinction between a subcritical and a supercritical vortex. For a vortex pair, Marcus & Berger [1] have studied time evolution by linear theory. However, their theory is incorrect for strong vortices: it is in conflict with the experimental, numerical and analytical results of Willmarth *et al.* [2], Telste (1989) and Tyvand [3]; these take the full nonlinearity into account and are in mutual agreement.

Peregrine: A comment: the initial condition $\Phi = 0$ on the free surface is justified since there is an impulsive start and the impulsive pressure at the surface is zero.

Schultz: You may be interested in examining the letters by Bernal & Kwon [4] (experimental), by Willmarth *et al.* [2] (numerical and linear theory) and by Bernal *et al.* [5].

Tyvand: These papers are related to my work, but none of them are concerned with the single-vortex problem presented here. Also, all published studies of nonlinear effects are purely numerical whereas I apply a perturbation technique. Just now, there is a rapidly growing interest in the nonlinear interaction between free vortices and a free surface. This field was almost unexplored a couple of years ago!

Tulin: Are there any exact solutions for free-surface flows with an embedded vortex? (I know of one: some Russian work on a soliton with a vortex inside it).

Tyvand: In the case of the soliton/vortex, the nonlinear interaction must be weak, and the vortex strength must be adjusted to the soliton height. I doubt that steady solutions are possible at all if there is strong nonlinear interaction between the free surface and a vortex moving freely according to Helmholtz's theorem.

References

[1] D.L. Marcus & S.A. Berger, 'The interaction between a counter-rotating vortex pair in vertical ascent and a free surface', *Phys. Fluids A*1 (1989) 1988–2000.

[2] W.W. Willmarth, G. Tryggvason, A. Hirsa & D. Yu, 'Vortex pair generation and interaction with a free surface', *Phys. Fluids A*1 (1989) 170–172.

[3] P.A. Tyvand, 'On the interaction between a strong vortex pair and a free surface', *Phys. Fluids*, submitted (1989).

[4] L.P. Bernal & J.T. Kwon, 'Vortex ring dynamics at a free surface', *Phys. Fluids A*1 (1989) 449–451.

[5] L.P. Bernal, A. Hirsa, J.T. Kwon & W.W. Willmarth, 'On the interaction of vortex rings and pairs with a free surface for varying amounts of surface active agent', *Phys. Fluids A*1 (1989) 2001–2004.