NUMERICAL COMPUTATION OF RESONANT STATES FOR THE 2-D LINEAR SEA-KEEPING PROBLEM

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Introduction

This study concerns fluid-structure interacting systems such as the behaviour of a ship under the action of swell, or the scattering of an acoustic wave by an elastic body. For such problems, one of the most important question is to work out the excitation frequencies which make maximum the amplitude of the motions of the structure: these frequencies are called "resonance frequencies". Let us stress that these are not eigenvalues as in classical conservative systems, for the amplitude of the response remains finite : this prevents use of classical methods for their computation.

In the case of a one-degree-of-freedom floating body, X.J. WU, Y. WANG and W.G. PRICE [1] show, by an asymptotic approach, how to compute these resonance frequencies: they search complex eigenvalues (i.e. complex excitation frequencies) of the system on the assumption that their imaginary parts are small, and give an expression which provides a first approximation of their real parts (i.e. of the resonant frequencies).

Our approach is somewhat different, although the basic idea is the same: we first define these complex eigenvalues, the "scattering frequencies", which are clearly identified as solutions of a nonlinear complex eigenvalue problem; the next step consists in expanding the solution of the problem in the vicinity of a scattering frequency in order to compute the location of the resonance and the associated amplitude of the solution. The method is explained here in the case of the 2-D sea-keeping problem for which we will present numerical results.

1. The sea-keeping problem: classical approach

Let us consider a rigid body (C) which floats (without forward motion) on the free surface of a fluid. When the system is at rest, the fluid occupies an unbounded domain $\Omega \subset \mathbb{R}^2$ whose boundary $\partial\Omega$ is defined by the "unperturbed free surface" (SL), the bottom (F) which is supposed to be rectilinear, and the hull (Γ) of the body (figure 1); n will denote the unit outward normal on $\partial\Omega$.

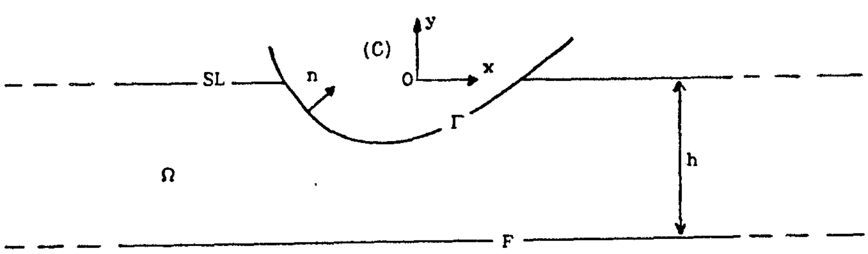

Figure 1

The axes (Ox) and (Oy) are defined as shown in figure 1. The fluid is assumed ideal; let Φ be the velocity potential function. The motions of the body around its equilibrium position will be described by the vector $U \in \mathbb{R}^3$ whose components characterize its 3 degrees of freedom (2 translations and 1 rotation).

We study the "linearized steady-state problem", i.e. the periodic motions of the system in the case of a sinusoidal incident wave of frequency ω. We are thus led to the following problem (expressed here in a non-dimensional form):

$$(\mathcal{P}_{\omega^2}) \begin{cases} (1) \quad \Delta \Phi = 0 \quad \text{in } \Omega \\ (2) \quad \partial_y \Phi - \omega^2 \Phi = 0 \quad \text{on SL} \\ (3) \quad \partial_y \Phi = 0 \quad \text{on F} \\ (4) \quad \partial_n \Phi + \omega\, U.\mathcal{N} = F_\Phi(\omega^2) \quad \text{on } \Gamma \\ (5) \quad (-\omega^2 \mathcal{M} + \mathcal{K})\, U + \omega \int_\Gamma \Phi\, \mathcal{N}\, d\gamma = F_U(\omega^2) \\ (6) \quad \int_{-h}^{0} \left| \dfrac{\partial \Phi}{\partial |x|} - i\nu_0 \Phi \right|^2 dy \xrightarrow[|x| \to \infty]{} 0 \qquad \text{(radiation condition)} \\ \qquad \text{where } \nu_0 \text{ is the positive solution of: } \operatorname{th}(\nu_0 h) = \dfrac{\omega^2}{\nu_0} \end{cases}$$

$\mathcal{M}$ is the 3×3 mass matrix of the body; $\mathcal{K}$ is the hydrostatic stiffness matrix. $\mathcal{N}$ is the "generalized normal" on every point X of the hull (Γ): $\mathcal{N}(X) = (n, OX \wedge n)$. F_Φ et F_U define the external forces (they depend on the incident swell).

It can be proved that this problem is well-posed except maybe for a sequence of frequencies ω_n (the existence of such eigenvalues of the problem seems unlikely but until now, no uniqueness theorem for the coupling problem has been proved). Different numerical methods can be implemented to solve this problem: the integral equation method, the coupling between finite elements and integral representation [2] or the localized finite element method ([2],[3]). We will use here the latter to describe our method.

Let $\widehat{\Omega} \subset \Omega$ be the bounded domain delimited by the two vertical segments Σ_1 and Σ_2 which are chosen such that $\Gamma \subset \partial\widehat{\Omega}$. $\widehat{SL}$ and $\widehat{F}$ refer to the parts of SL and F which are contained in $\partial\widehat{\Omega}$; $\breve{\Omega}_\ell$, $\breve{SL}_\ell$, $\breve{F}_\ell$ (for ℓ=1,2) denote respectively the

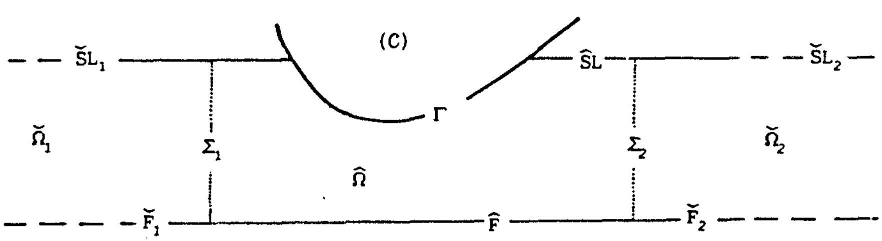

Figure 2

left and right parts of $\Omega\backslash\widehat{\Omega}$, $SL\backslash\widehat{SL}$ and $F\backslash\widehat{F}$ (see figure 2).

Let us consider, for $\ell=1,2$, the operator $C_\ell(\omega^2)$ which associates to a given function χ defined on Σ_ℓ the normal derivative of the solution of the "exterior problem", i.e. the potentiel function $\breve{\Phi}_\ell$ which satisfies : (1) in $\breve{\Omega}_\ell$, (2) on $\breve{SL}_\ell$, (3) on $\breve{F}_\ell$, the radiation condition (6) and $\breve{\Phi}_\ell = \chi$ on Σ_ℓ. By the method of separation of variables, $C_\ell(\omega^2)$ can be expressed as a convergent series (which will be truncated for numerical applications). One can easily prove that the solution $(\widehat{\Phi},\widehat{U})$ of the new problem (set in the bounded domain $\widehat{\Omega}$):

$$(\widehat{\mathcal{P}}_{\omega^2})\begin{cases}(1)\text{ in }\widehat{\Omega}\ ;\ (2)\text{ on }\widehat{SL}\ ;\ (3)\text{ on }\widehat{F}\ ;\ (4)\text{ on }\Gamma\ ;\ (5)\\ (6)'\quad \partial_n\widehat{\Phi} = -C_\ell(\omega^2)\left(\widehat{\Phi}|_{\Sigma_\ell}\right)\quad\text{on }\Sigma_\ell\quad(\ell=1,2)\end{cases}$$

is such that $\widehat{\Phi}$ is the restriction of the solution Φ of $(\mathcal{P}_{\omega^2})$ and $\widehat{U} = U$. Under variational form, problem $(\widehat{\mathcal{P}}_{\omega^2})$ writes as:

$$(\widehat{\mathcal{P}}^*_{\omega^2})\qquad \text{Find}\quad (\widehat{\Phi},\widehat{U}) \in H^1(\widehat{\Omega}) \times \mathbb{C}^3\quad\text{such that :}\quad \widehat{S}(\omega^2)(\widehat{\Phi},\widehat{U}) = \mathcal{F}(\omega^2)$$

where $\mathcal{F}$ depends linearly on F_Φ and F_U, and $\widehat{S}(\omega^2)$ is a linear operator which depends continuously on ω^2 (see §2). Problem $(\widehat{\mathcal{P}}^*_{\omega^2})$ is solved by finite elements. For each excitation frequency ω in a given frequency range, we can then compute the total energy of the body, which defines the "response curve" of the body: our purpose consists in locating the maxima of this curve (the "resonant states") without computing the whole response curve.

2. Scattering frequencies of the coupled system

The basic idea of the method lies in the extension of problem $(\widehat{\mathcal{P}}_{\omega^2})$ to the case of complex frequencies. We first construct explicitly the analytical continuation of $C_\ell(\omega^2)$ (which will be denoted by $C_\ell(\nu)$, $\nu\in\mathbb{C}$) in the complex plane. Therefore the "extended problem" $(\widehat{\mathcal{P}}_\nu)$ formally obtained by replacing ω^2 by any complex number ν makes sense; its variational formulation writes as in $(\widehat{\mathcal{P}}^*_{\omega^2})$, where $\widehat{S}(\nu)$ is now defined for complex numbers.

If ν has a positive imaginary part, then problem $(\widehat{\mathcal{P}}_\nu)$ is always well-posed and its solution tends to the solution of $(\widehat{\mathcal{P}}_{\omega^2})$ when ν tends to ω^2. The operator $\widehat{R}(\nu) = (\widehat{S}(\nu))^{-1}$ (which associates to the data $\mathcal{F}(\nu)$ the solution of $(\widehat{\mathcal{P}}_\nu)$) is an analytical function of ν if $\Im m(\nu) > 0$. One can prove that it has a meromorphic continuation in the lower half complex plane. In other worlds, if $\Im m(\nu) < 0$, problem $(\widehat{\mathcal{P}}_\nu)$ is well-posed except on isolated singularities which are poles of $\widehat{R}(\nu)$; these poles are the "scattering frequencies" of the coupled system : they are solutions of the following non-linear eigenvalue problem:

$$\int_{\hat{\Omega}} \nabla\hat{\Phi}.\overline{\nabla\hat{\Psi}}\, dx + \hat{V}^{*} \mathcal{K} \hat{U} - \nu \int_{\hat{S}L} \hat{\Phi}\, \overline{\hat{\Psi}}\, dx - \nu \hat{V}^{*} \mathcal{M} \hat{U} + \sqrt{\nu} \int_{\Gamma} \left[\hat{U}.\mathcal{N}\, \overline{\hat{\Psi}} + \hat{\Phi}\, \overline{\hat{V}.\mathcal{N}}\right] d\gamma$$

$$+ \sum_{\ell=1,2} \int_{\Sigma_\ell} C_\ell(\nu) \left(\hat{\Phi}|_{\Sigma_\ell}\right) \overline{\hat{\Psi}}\, d\sigma = 0 \quad ; \quad \forall\, (\hat{\Psi},\hat{V}) \in H^1(\hat{\Omega}) \times \mathbb{C}^3$$

3. Approximation of resonant states

The discretization (by finite elements) of $(\hat{\mathcal{P}}_\nu^{*})$ leads us to a finite dimensional problem : $\mathbb{S}(\nu)\ X = F(\nu)$, where $\mathbb{S}(\nu)$ is a holomorphic N×N complex matrix family. An approximate scattering frequency $\tilde{\nu}$ will thus be solution of the non-linear matrix eigenvalue problem : $\mathbb{S}(\tilde{\nu})\ X = 0$ (which is equivalent to say that 0 is an eigenvalue of $\mathbb{S}(\tilde{\nu})$). This latter problem will be solved by an iterative method (such as the fixed point or the Newton method).

The perturbation theory for linear operators (KATO [4]) allows us to construct explicitly the expansion of $\mathbb{R}(\nu) = (\mathbb{S}(\nu))^{-1}$ in the vicinity of a scattering frequency $\tilde{\nu}$: if $\tilde{\nu}$ is close to the positive real axis, this series will naturally provide the one of the solution of the steady-state problem which writes : $X = \mathbb{R}(\omega^2)\ F(\omega^2)$. For example, if $\tilde{\nu}$ is simple (i.e. if 0 is a simple eigenvalue of $\mathbb{S}(\tilde{\nu})$), one obtains:

$$X(\omega^2) = \frac{1}{\omega^2 - \tilde{\nu}}\ \frac{\tilde{Y}^{*} F^{(0)}}{\tilde{Y}^{*}\mathbb{S}^{(1)}\tilde{X}}\ \tilde{X} + O(1) \quad \text{where} \quad \begin{cases} \mathbb{S}(\nu) = \mathbb{S}(\tilde{\nu}) + (\nu-\tilde{\nu})\mathbb{S}^{(1)} + O(\nu-\tilde{\nu})^2 \\ F(\nu) = F^{(0)} + O(\nu-\tilde{\nu}) \end{cases}$$

$\tilde{X}$ and $\tilde{Y}$ are respectively the right and left eigenvectors of $\mathbb{S}(\tilde{\nu})$ (chosen such that: $\tilde{Y}^{*}\ \tilde{X} = 1$). This development of $X(\omega^2)$ expresses the influence of a complex singularity of the problem upon the response of the system for real frequencies : it clearly shows that the amplitude of the response will have a maximum in the vicinity of the scattering frequency $\tilde{\nu}$. We thus locate the resonant states of the coupled system.

References

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