NON-LINEAR LONG INTERFACIAL WAVES DUE TO A DISTURBANCE MOVING IN A THIN THERMOCLINE

T. Miloh
School of Engineering,
University of Tel-Aviv, Israel

and

M.P. Tulin
Ocean Engineering Laboratory,
University of California,
Santa Barbara, CA, U.S.A.

It is known that oceans and lakes often exhibit a well-mixed warm surface layer separated from a colder homogeneous region by a shallow *thermocline*, or a layer of large density gradient. The relative density change across the thermocline may vary according to formation circumstances, but it is usually of the order of 10^{-3} in open seas and 3×10^{-2} in the Norwegian fjords and ice-rimmed seas: The density stratification may be due to temperature or salt-content variation and, in fjords for example, can be enhanced by a seaward flow of fresh water in the upper layer. In such cases the Brunt-Väisälä frequency peaks sharply in the vicinity of the thin thermocline and is small elsewhere. In the 'two-layer' model, for example, the Brunt-Väisälä may be approximated by a delta function $\delta(z-h)$, where h denotes the depth of the upper homogeneous stable layer of density ρ_1 lying above an heavier homogeneous layer of density ρ_2 and depth H.

We consider a general disturbance which moves steadily on or below the free-surface. This may be due to a surface ship, pressure patch, free-surface impulse or an arbitrary submerged singularity system simulating a flow induced by submerged body or fixed structure. If the characteristic wave length of the induced wavy motion, is assumed to be small with respect to the thermocline depth, the quadratic dispersion relationship yields two distinct values (modes). The first is the 'barotropic' mode, which is identical with a surface wave on a layer of constant density with depth h+H. The second 'baroclinic' mode is an internal mode moving with pase velocity given by $C^2 = gh(\rho_2/\rho_1 - 1)$ and which attains a maximum amplitude on the interface. The internal wave may also induce a relatively large horizontal velocity near the free-surface, with regions of maximum convergence of the horizontal velocity lying just above the nodes of the interface displacement. The resulting gathering of contaminants on the free-surface can produce visible lines or 'slicks' which usually indicate the presence of sub-surface internal waves. The free-surface vertical displacement due to the 'barotropic' mode is generally much smaller by a factor $\rho_2/\rho_1 - 1$, than that induced by the 'baroclinic' mode and thus, for many practical applications, the free surface may be replaced by a 'rigid-lid', as far as internal waves are concerned. Hence, since very small changes in potential energy are actually needed to produce internal waves, an imposed moving disturbance whose free-surface signature may be hardly noticed, can be very effective in generating large amplitude internal waves. This indeed was the explanation suggested by Bjerknes for the so called "dead water" phenomenon, where slow vessels were found to experience a sudden large increase in resistance at low speed in shallow thermoclines. In addition to the increase in wave drag and a subsequent loss in steerage, Ekman (1904) also reported on some striking wave patterns which were observed on the free-surface. These include sharp bands of disturbance propagating nearly transversely (to the ship direction) and extending great distances to the side of the ship. Such a wave pattern resembles somewhat the shock pattern around an airfoil at transonic speeds. 'Dead water' phenomena usually occur in the vicinity of the critical densimeter Froude number, e.g. F=1, where F=U/C, and U is the disturbance (or flow) speed. It is also important to note that near the critical densimeter Froude number the linearized theory ceased to be valid and both non-linear as well as dispersive terms must be included in the formulation in a consistent manner. The problem of wave resistance in 'dead water' situation has been recently discussed by Miloh & Tulin (1988), and now we intend to concentrate on the explanation of the peculiar free-surface pattern observed at critical conditions.

In the first part of the paper we present a general non-linear formulation in the wave-number space for the elevation of a 'two-layer' interface due to an arbitrary forcing. We employ the 'rigid-lid' approximation and show that in two-dimensional flows the resulting non-linear evolution equations, reduce to the Korteweg-de Vries (KdV) equation when the depths of the two-layers are of the same order of magnitude. In the case where the lower layer is infinitely deep the Benjamin-Ono (BO) equation is recovered in the form of a non-linear singular integro-differential equation where the dispersive term is given by the Hilbert transform. For 'intermediate' depths of the lower layer, one obtains the so-called ILW (intermediate long wave) equation, which may be considered as a 'finite depth' generalization of both the KdV and BO equations. For three-dimensional flows it is demonstrated, how the 2-D model equations may be extended in the manner of Kadomtsev-Petviashvilli, (KP).

It is claimed that at least some of the observed free-surface signatures are manifestations of interfacial , or periodic solitary waves, which, as a result of a balance between non-linearity and dispersive effects, can propagate without a considerable change in shape. Solitons do appear only at supercritical speeds (U>C), whereas periodic solitary waves may appear at subsonic speeds as well. A sequence of steep asymmetric periodic waves, which are peaked upward (similar to cnoidal waves), were indeed observed in some of the mariner sketches in Ekman's report. For this reason special effort has been directed towards the calculation of periodic stationary wave solution of the 2-dimensional ILW equation. It was very rewarding to find a closed form analytic expressions for these non-linear periodic waves in the form of convergent Fourier series with prescribed coefficients. What was even more surprising, was the fact that these periodic solutions may be represented as an infinite sum of equally spaced identical solitons. Thus, ILW solitons may be added together *linearly* to render a periodic solution of a *non-linear* problem. A similar property has been recently found for the KdV (Whitham (1984)) and for the BO equation (Miloh & Tulin (1989)), which may be considered as degenerate cases of the more general ILW equation. For the three-dimensional (KP) version of the ILW equation it is also possible to obtain analytic expressions for a plane-wave soliton or periodic wave-train, which propagate nearly transversely to the ship's direction and extending great distances to the sides of the ship.

Finally, we present a non-linear calculation of the wave drag and the interfacial disturbance due to a submerged two-dimensional dipole moving steadily above or below the interface. The particular choice of a dipole enables us to obtain analytic solutions for the wave resistance in the vicinity of the critical densimenter Froude number (including effects of weakly non-linear terms). With this solution at hand it is now possible to discuss the relative importance of non-linear inertial to linear dispersive terms in various regions of Fourde numbers. The maximum wave drag of the dipole was found to be shifted downwards from the critical region towards subsonic speeds. The amount of the shift is proportional to the ratio between the square root of the dipole output and the submergence depth. The interfacial disturbance was found to be much larger when the dipole was above the interface within the thin thermocline compared to the situation where the dipole was beneath the interface. A proper scaling of the governing weakly non-linear equations also determine the range of parameters for which these equations reduce to the Michell-Stretenski shallow water equation, or to the forced Prandtl-Glauret equations in gas dynamics, which admit shocks and transonic drag.

References

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