ON THE COMPUTATION AND EXCITATION OF TRAPPING MODES

by

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Consider a long, horizontal cylinder, submerged beneath the free-surface of deep water. Such a configuration can support trapping modes. In order to describe these, we begin by choosing Cartesian coordinates (x,y,z) so that $y = 0$ is the mean free surface, the z axis is parallel to the generators of the cylinder, and y increases with depth. Now, look for a potential Φ that satisfies Laplace's equation in the water, $\partial\Phi/\partial n = 0$ on the cylinder, and the linearized free-surface condition, $K\Phi + \partial\Phi/\partial y = 0$ on $y = 0$ (where $K = \omega^2/g$ and ω is the radian frequency of oscillation); in addition, Φ is assumed to be a trigonometric function of kz, where k is a real constant.

If $0 \leq k < K$, this problem arises in the scattering of a plane wave by the cylinder : Φ is then the total potential, and $k = K\cos\alpha$, where α is the angle between the direction of propagation of the incident wave and the z-axis.

If $k > K > 0$, there are non-trivial, bounded solutions Φ, but only at discrete frequencies, i.e. if k is fixed ($>K$), there is at least one value of K for which Φ is not identically zero ([8], Theorem 5.2). This result holds for cylinders of any cross-section. It is also known that Φ decays exponentially as $|x| \to \infty$ [7]. These potentials are called trapping modes.

I became interested in trapping modes in 1984, for two reasons. First, I was intrigued by a parenthetical remark in Ursell's paper [6] in which he constructs a symmetric (even function of

x) trapping mode for a circular cylinder centred at $x = 0$, $y = f$: '...antisymmetrical trapping modes cannot be constructed by the present method' (p.350). This encouraged me to look for such modes, numerically. A boundary integral equation of the second kind (with a continuous kernel) was derived, using Green's theorem. After discretization, the determinant of the corresponding matrix was computed, and then zeros were sought. The program was validated by comparing with the numerical results of McIver and Evans [5] (they implemented Ursell's method [6]). Antisymmetric modes were also found above a circular cylinder. Similar results were found for elliptic cylinders. Trapping modes were also found above rotated elliptic cylinders (so that their cross-sections were not symmetric about $x = 0$).

Second, I examined a Ph.D. thesis by A.D. Burden on the propagation and excitation of elastic surface waves along a cylindrical cavity in an otherwise unbounded elastic solid; a finite number of modes always exists [2][3]. These waves are analogous to Rayleigh waves on an elastic half-space. The excitation of Rayleigh waves by a point force is known as Lamb's problem, after Lamb's famous paper of 1904 [4]. It occurred to me that similar methods should work for the excitation of trapping modes by a point wave-source, near the submerged cylinder. This would provide a linear mechanism for their excitation (clearly, trapping modes cannot be excited by scattering a plane wave from infinity), rather than the nonlinear mechanisms suggested by other authors [1].

Some results of the calculations and computations described above will be presented at the Workshop.

References

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[3] Burden, A.D. The propagation of elastic surface waves along cylindrical cavities of general cross section. Wave Motion 7 (1985) 153-168.

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DISCUSSION

Kleinman: Why doesn't the existence of trapping models violate the Maz'ja uniqueness theorem for submerged bodies?

Martin: The water-wave problem is a 3-D problem, with an infinitely long submerged cylindrical body; such unbounded bodies are not allowed by Maz'ja's theorem.

Newman: Could you explain "Rayleigh's hypothesis" and how it relates, for example, to the field outside an ellipse?

Martin: Consider acoustic scattering (Helmholtz equation) by a cylinder in 2-D. For a circular cylinder, we can use separation of variables in polar coordinates; this gives

$$\phi = \Sigma_n A_n H_n^{(1)}(kr)e^{in\theta}\,, \qquad (*)$$

where A_n can be found using the boundary condition on the cylinder. The Rayleigh hypothesis (RH) says that ϕ can be expanded as (*) for cylinders of *any* cross-section. In general, this is false. It is known that the RH is true for *some* smooth cross-sections. Given any particular cross-section, there are methods for determining whether the RH is valid, or not; see e.g. van den Berg & Fokkema, 1979. In the case of an ellipse, with centre at $r=0$, foci at $x=\pm c$ and semi-minor axis of length b, the conclusion is as follows; the RH is valid if $b\geq c$ and is false if $b<c$.

For a review of the RH (old, but instructive!), see the paper by R.F. Millar in Radio Science, 1973.

Tuck: 1) Are the anti-symmetric eigenfrequencies interleaved with the symmetric ones?
2) Is the anti-symmetric multiple expansion complete, in view of the fact that its $n=0$ term is (unlike one symmetric case) not source-like?

Martin: 1) From my limited numerical results, it appears that they are.
2) Yes. The $n=0$ term is absent. The $n=1$ term corresponds to a horizontal dipole, which (unlike a source) is anti-symmetric.

Ursell: We might alternatively consider a transient line source near the cylinder. Some of the energy would go into the continuous spectrum and radiate to infinity, the remainder would go into the trapping modes which then continue to oscillate without decay.

Martin: A time-harmonic point source excites all values of k for a fixed value of K. I think that your alternative would also work; it corresponds to a fixed value of k and all values of K (through a Fourier decomposition in time of the transient source).