# EVALUATION OF THE WAVE-RESISTANCE GREEN FUNCTION NEAR THE SINGULAR AXIS

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The potential of a source moving with constant horizontal velocity in a fluid of infinite depth is fundamental to the linearized analysis of ship waves, including both the classical thin-ship approximation, and more comprehensive boundary-integral solutions of the Neumann-Kelvin problem. Despite the practical importance of this topic, and the great attention it has received in the past century, effective methods for evaluating the source potential exist only for part of the physical domain of interest.

The potential considered here is the harmonic function which must be added to $1/r$ to satisfy the free-surface and far-field conditions. The origin is taken at the position of the image source above the free surface, with $x$ positive in the direction of forward motion and $z$ positive downwards; the coordinates are nondimensionalized by gravity and the forward velocity. Circular cylindrical coordinates $(x, \rho, \alpha)$ are used, with $z + iy = \rho e^{i\alpha}$. The potential is an even function of $\alpha$ and we shall assume here that this angle is positive. Note that the limit $\alpha = \pi/2$ is the plane $z = 0$.

An effective form of the source potential for numerical work is the sum of a double integral which is an even function of $x$, and a single integral which is included only for $x < 0$. At the First Workshop two years ago I showed that the double integral defined in this manner can be efficiently evaluated by polynomial expansions for all points in the physical domain. (The coefficients for these expansions and more details are given in [1].) It remains only to consider the single integral defined by

$$f(x,y,z) = \int_1^\infty \frac{s}{\sqrt{s^2-1}} e^{-zs^2} \sin(xs) \cos(ys\sqrt{s^2-1})\, ds \tag{1}$$

Two very useful Neumann expansions given by Bessho ([2], equations 4.2-3) provide efficient algorithms for the single integral in certain domains. The first, (4.2), is uniformly convergent for all finite values of the parameter $M \equiv x^2/(4\rho)$, but the convergence is slow for large values of this parameter and the accuracy is reduced by the accumulation of cancellation errors. The complementary expansion (4.3) is asymptotically convergent, and can be used to achieve 5-6 decimals accuracy if $M \geq O(100)$, provided $\alpha < \pi/2 - \delta$ where $\delta$ is a small positive number, the value of which depends on $M$ and on the accuracy required. The use of these expansions is illustrated by Baar and Price [3].

Some improvement in the convergence of (1) and in the two complementary Bessho expansions can be achieved by associating the dominant singularity of the integral in (1) with Dawson's integral ([5], p. 298). The following analysis extends that given in [4] for the special case $y = 0$.

To improve the convergence of (1) for $s \to \infty$ it is appropriate to subtract the simpler integral

$$\int_0^\infty e^{-zs^2} \sin(xs) \cos\left(y(s^2 - \tfrac{1}{2})\right) ds = \mathrm{Re}\{e^{iy/2}\left(\frac{2\xi}{x}\right) F(\xi)\} \tag{2}$$

where $F$ is Dawson's integral and $\xi = M^{1/2}e^{-i\alpha/2}$. Note that the difference between the integrands of (1) and (2) is $O(s^{-2})$ for all values of $x, y, z$.

We are particularly concerned with the case $M >> 1$. For large values of its argument Dawson's integral can be expanded in terms of the complementary error function, with the result

$$F \simeq -\frac{\sqrt{\pi}}{2} ie^{-\xi^2} + \frac{1}{2\xi}\left(1 + \sum_{m=1} \frac{1 \cdot 3 \cdot \ldots (2m-1)}{(2\xi^2)^m}\right) \tag{3}$$

The first term in this expansion, which is exponentially small except near $\alpha = \pi/2$, gives the following approximation to (2):

$$\int_0^\infty e^{-zs^2} \sin(xs) \cos\left(y(s^2 - \tfrac{1}{2})\right) ds \simeq -\tfrac{1}{2}\left(\frac{\pi}{\rho}\right)^{1/2} \mathrm{Re}\left[i \exp\left(iy/2 - i\alpha/2 - Me^{i\alpha}\right)\right] + O(1/\xi) \tag{4}$$

Since the difference between (1) and (2) is bounded as $\rho \to 0$ it can be inferred that (4) represents the leading-order singularity of the single integral near the axis $\rho = 0$. (A simplified analysis with analogous results is given by Euvrard [6].) While useful for deriving the leading-order singularity on the axis, this *ad hoc* approximation cannot be extended to higher order for computational purposes.

The singularity (4), associated with short diverging waves of vanishingly small wavelength, was noted by Ursell [7] in the closely-related solution for a concentrated pressure on the free surface. In the simplest terms it is a result of the fact that the single integral (1) does not converge when $\rho = 0$. This singularity is absent from Bessho's asymptotic expansion, which tends smoothly to the regular limit $f(x,0,0) = -(\pi/2)Y_1(x)$ for all values of $\alpha$.

Following informal discussions of this topic at the Second Workshop, Ursell [8] has derived a more complete and rigorous asymptotic expansion for large $M$, including Bessho's equation 4.3 together with an additional integral which is exponentially small except near $\alpha = \pi/2$. An auxiliary function $F$ is defined (not to be confused with the Dawson integral above), from which (1) may be evaluated by differentiation:

$$f(x,y,z) = -\tfrac{1}{2}e^{-z/2}\frac{\partial F}{\partial x} \tag{5}$$

Ursell shows for large $M$ that

$$F \simeq -\pi I_0(\tfrac{1}{2}\rho)Y_0(x) - 2\pi \sum_{m=1} I_m(\tfrac{1}{2}\rho)Y_{2m}(x)\cos(m\alpha) - iUe^{i\alpha/2}\exp(-Me^{-i\alpha}) \tag{6}$$

where

$$U = \int_{-\infty}^{\infty} e^{-M\sigma^2} \exp\left[\frac{\rho^2 e^{i\alpha}}{16M(1 - i\sigma e^{i\alpha/2})^2} + \frac{\rho e^{i\alpha}}{2(1 - i\sigma e^{i\alpha/2})}\right] \frac{d\sigma}{(1 - i\sigma e^{i\alpha/2})} \tag{7}$$

The Neumann series involving the Bessel functions $I_n, Y_n$ in (6) corresponds to Bessho's equation 4.3; the additional contribution derived by Ursell, involving the integral (7), renders the approximation (6) asymptotic with an exponentially small error for all values of $\alpha$ up to and including the plane $\alpha = \pi/2$. Ursell also presents a one-term steepest-descent approximation for (7), and differentiation of that result with respect to $x$ gives precisely the same leading-order result displayed in (4).

For numerical purposes a a more complete asymptotic expansion of (7) is required. This can be derived by expanding the integrand (excluding the first exponential) in powers of $\sigma$, and integrating term-by-term. An algorithm for this procedure is as follows:

1. Denote the sum in brackets in (7) by $g(\sigma)$ and expand the difference

$$g(\sigma) - g(0) \simeq \sum_{n=1}^{2N} A_n^{(1)} \sigma^n$$

in a truncated Taylor series, with the coefficients

$$A_n^{(1)} = -\left[\frac{\rho^2(n+1)}{16M} + \tfrac{1}{2}\rho\right]\left(ie^{i\alpha/2}\right)^{n+2} \qquad (n = 1, 2, ..., 2N)$$

2. Evaluate the coefficients $A_n^{(m)}$ in the corresponding series

$$\left[g(\sigma) - g(0)\right]^m \simeq \sum_{n=1}^{2N-m+1} A_n^{(m)} \sigma^{n+m-1}$$

from the recursion

$$A_n^{(m+1)} = \sum_{k=1}^{n} A_k^{(m)} A_{n-k+1}^{(1)} \qquad (n = 1, 2, ..., 2N - m)$$

3. Evaluate the coefficients $B_n$ in the expansion

$$\exp\left(g(\sigma) - g(0)\right) \simeq 1 + \sum_{n=1}^{2N} B_n \sigma^n$$

from the relation

$$B_n = \sum_{m=1}^{n} \frac{1}{m!} A_{n+1-m}^{(m)} \qquad (n = 1, 2, ..., 2N)$$

4. Expand the last denominator of (7) in a Taylor series with coefficients $\left(ie^{i\alpha/2}\right)^n$
5. Form the product of the expansions in steps 3 and 4 above,

$$\frac{\exp\left(g(\sigma) - g(0)\right)}{\left(1 - i\sigma e^{i\alpha/2}\right)} \simeq 1 + \sum_{n=1}^{2N} C_n \sigma^n$$

where

$$C_n = \left(ie^{i\alpha/2}\right)^n + \sum_{m=1}^{n} B_m \left(ie^{i\alpha/2}\right)^{n-m}$$

The asymptotic expansion of (7) follows immediately in the form

$$U \simeq \left(\frac{\pi}{M}\right)^{1/2} e^{g(0)} \left(1 + \sum_{k=1}^{N} C_{2k} \frac{1 \cdot 3 \cdot \ldots (2k-1)}{(2M)^k}\right) \qquad (8)$$

where

$$g(0) = \left(\rho^2/16M + \rho/2\right)e^{i\alpha}$$

This algorithm can be coded in three nested loops if the triangular matrix $A_n^{(m)}$ is evaluated along each diagonal $n + m =$ constant. Apart from a factor $(ie^{i\alpha/2})^{n-1+3m}$ the elements of this matrix are real. Spatial derivatives involving higher powers of the last denominator can be obtained by repeating the last recursion.

Numerical experiments indicate that (8) converges to about five significant decimals accuracy if $M \geq 10$ and $x \leq 1$, but for larger values of $x$ the useful domain is reduced. It remains to find effective algorithms for the single integral in the far-field where both $x$ and $M = x^2/4\rho$ are large. This domain includes the neighborhood of the $19\frac{1}{2}^\circ$ cusp line.

Another important issue concerns the appropriate numerical analysis of contributions from the source potential in the domain where the expansions above are valid; in a typical boundary-integral formulation these components would be integrated over discretized panels on the ship hull surface, and the proper treatment of such highly-oscillatory influence functions is not clear. Integration of oscillatory wave-like components can be performed analytically over quadrilateral panels in the manner described by Doctors and Beck [9], but this only improves the accuracy by one or two algebraic orders at the expense of using fast algorithms to evaluate the source potential itself. This problem may be more critical for the infamous line integral of the Neumann-Kelvin theory, over the intersection of the ship hull and the free surface. If the line integral is discretized, for example in a sequence of straight segments, inaccuracies are inevitable in the solution along rays extending downstream from the discontinuities between adjacent segments. Even if this line integral is integrated exactly, one must expect singular contributions for the end points at the bow and stern!

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**Maruo**: The asymptotic behavior of the Kelvin source along its track is disclosed by the expansion employed in the slender-body theory. Assume that $y$ and $z$ are $O(\epsilon)$, and $x$ is $O(1)$. The asymptotic expression of the Kelvin source is that given by Tuck in 1963. This is the principal term of Bessho's expansion. However, this expansion does not hold near the singularity.

If $x$ is small, of $O(\epsilon^{1/2})$, the Kelvin source has an asymptotic expression which can be reduced to a known function, *i.e.* the error function of imaginary argument, and the Bessho type expansion is not valid. A uniformly valid asymptotic expression can be obtained by means of matching in the far field, and includes both types of expressions.

Reference: 10th Weinblum Memorial Lecture, "Evolution of the Theory of Slender Ships", *Schiffstechnik.* To be published.