Evolution of Nonlinear Waves Due to a Moving Wall

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Nonlinear transient waves induced by the movement of a wall have been studied. It is known that analytical solutions can become singular at the intersection of the free surface with the wall due to a confluence of the boundary conditions (e.g., Lin, 1984). This can have global influence on the computation and causes fundamental difficulties (Dommermuth & Yue, 1986). We, however, explore the problem more carefully and conclude that the singularity depends on the analytical method applied and the computational difficulty can be resolved with less effort than expected.

When the wall moves with constant acceleration, we apply small-time asymptotics to the inviscid and incompressible model to obtain a solution which satisfies all the initial and boundary conditions except at the contact line. Chwang (1983) argues that the singularity is outside of the physical domain by using a stationary coordinate system and expanding the wall boundary condition about its initial location. But that expansion is made about the singularity, which makes his straightforward expansion invalid. Formulation of the problem in a Lagrangian or moving coordinate system, which obviates the expansion of the boundary condition, reveals that the singularity is at the contact line. This is confirmed by checking conservation of mass and energy. The contact-line singularity is shown to be caused by the use of the small-time expansion. The independent variables are coupled, so that any method which separates time from the spatial variables, such as the small-time expansion, cannot approximate the solution properly. In fact, we show that the resulting solutions fail to be asymptotic in time within a distance $O(t^2)$ of the contact line. We naturally seek a self-similar correction, which cancels the logarithmic singularity at the contact line and becomes of higher order when $z(=x+iy) \gg t^2$, where the origin of the coordinate system is at the undisturbed contact line. For small Froude number, we can obtain a closed-form solution for this self-similar correction, which enables us to write the complete solutions as

$$u - iv = i\frac{2\alpha t}{\pi}[\ln \tanh \frac{\pi z}{4} + \int_0^\infty (\frac{1}{k} - \frac{\sin\sqrt{k}}{k^{3/2}})e^{-ikz/t^2} dk]$$

$$\eta = -\frac{\alpha t^2}{\pi}[\ln \tanh \frac{\pi x}{4} + 2\int_0^\infty (\frac{1}{2k} - \frac{1-\cos\sqrt{k}}{k^2}) \cos \frac{kx}{t^2} dk]$$

where α is the Froude number and η is the free-surface elevation. Asymptotic evaluation of the above integrals near the contact line ($z \ll t^2$) gives the local solution as follows:

$$\begin{aligned} u - iv &= \alpha t + i\frac{2\alpha t}{\pi}(ln\frac{\pi t^2}{4} + \gamma - 2) - \\ &\quad \frac{4\alpha t}{\pi}[\frac{z}{t^2} + i(\frac{z}{t^2})^2 + \cdots] + [-\frac{8\alpha t}{\sqrt{\pi}}(\frac{z}{t^2})^{3/2} e^{i(\frac{t^2}{4z} - \frac{\pi}{4})} + \cdots] \end{aligned}$$

$$\eta = -\frac{2\alpha}{\pi}[\frac{t^2}{2}(ln\frac{\pi t^2}{4} + \gamma - 3) + \pi t^2(\frac{x}{t^2}) + 2t^2(\frac{x}{t^2})^2 + \cdots] - \frac{16\alpha t^2}{\sqrt{\pi}}(\frac{x}{t^2})^{5/2} cos(\frac{t^2}{4x} - \frac{\pi}{4}) + \cdots$$

These expansions agree with those of Roberts (1987), obtained by a more difficult method, except that the constant $\pi/4$ in the logarithmic terms is different because a somewhat different problem has been studied.

Based on the above analysis, we can also resolve some of the computational difficulties by incorporating the time dependence properly with the spatial variables through a modified time-marching method. The dynamic free-surface condition needs to be expanded about the initial ($\eta = 0$) and present ($\eta = \eta_0$) location of the free surface in the small-time expansion and time-marching process, respectively. That gives

$$\phi_t + (\eta - \eta_0)\phi_{ty} + \frac{(\eta - \eta_0)^2}{2!}\phi_{tyy} + \cdots + \frac{1}{2}\{[\phi_x + (\eta - \eta_0)\phi_{xy} +$$

$$\frac{(\eta - \eta_0)^2}{2!}\phi_{xyy} + \cdots]^2 + [\phi_y + (\eta - \eta_0)\phi_{yy} + \frac{(\eta - \eta_0)^2}{2!}\phi_{yyy} + \cdots]^2\} + \eta = 0$$

which is evaluated at $y = \eta_0$. In the small-time expansion, all the terms except the first one are of higher order. A posterori, some neglected terms become infinitely large near the contact line. Therefore, for small Froude number, we approximate $\eta - \eta_0$ by a small parameter, ϵ, which depends on time, and ignore the nonlinear terms. The resulting set of governing equations with a free-surface boundary condition of the Robin type yields a well-behaved solution even at the contact line. In this way, we satisfy the boundary conditions at the proper location of the free surface ($y = \eta$) rather than where it *was*, and the time dependence is carried by the desingularization parameter ϵ. When this is applied to the standard time-marching process, we first guess $\epsilon(x,t)$, which instantaneously gives us ϵ_t. Then, we can use

$$(\phi + \epsilon\phi_y)_t = -\frac{1}{2}(\phi_x^2 + \phi_y^2) - y - \eta_t\phi_y$$

to obtain the free-surface condition at next time step. The Neumann conditions on the wall and bottom are always known, so that we can solve for the flow field and the free-surface elevation. We, then, compare the increment of the elevation during the time step with the ϵ we guessed and iterate to convergence.

When the wall starts to move impulsively (Lin, 1984), the wall boundary condition is inconsistant with the initial state of the fluid while the field equation does not contain any time operator. Therefore, in order to pose the problem properly, we have to determine the change that the impulse has done to the flow field to obtain the initial conditions at $t = 0^+$. Since the Laplace equation does not require initial conditions, we can integrate the free-surface boundary conditions from $t = 0^-$ to $t = 0^+$ to get the initial conditions on the free surface, which are

$$\phi = 0, \qquad \eta = 0.$$

This time, the solution technique we used for the case of constant acceleration is not satisfactory. For sufficiently small time, we cannot linearize the free surface conditions for small Froude number. Once we keep the nonlinear terms, a similarity transformation no longer exists due to the constant-velocity condition on the wall. This nonlinear problem and a proper initial condition at $t = 0^+$ near the contact line are not yet fully understood.

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