THE EXCITING FORCE AND MOMENT IN LONG WAVES

by J. N. Newman

MIT, Cambridge, Mass., USA

In the prediction of wave-body interactions it is possible to simplify the analysis of the exciting force by making a long-wavelength or low-frequency approximation, which is applicable if the pertinent body dimensions are small compared to the wavelength. Morison's formula is an important example where, in the absence of viscous drag, the exciting force is approximated by the product of the incident-wave orbital acceleration and an inertial coefficient equal to the sum of the fluid mass displaced by the body and its added mass in an unbounded fluid. The basis for this approximation is the relation

$$F_i \simeq (m_{ij} + \rho \forall \delta_{ij}) \frac{\partial}{\partial x_j} \frac{\partial \Phi}{\partial t} + (m_{jk} + \rho \forall \delta_{jk}) u_k \frac{\partial}{\partial x_i} u_j \tag{1}$$

for the force component acting on a stationary body in the direction i, due to any external flow field with potential $\Phi(\mathbf{x}, t)$ and velocity components u_j. The fundamental assumption is that $\Phi(\mathbf{x}, t)$ is slowly-varying in space, relative to the length-scale of the body. In (1) the added-mass coefficients are denoted by m_{ij} and $\forall$ is the fluid volume displaced by the body. In two dimensions $\forall$ is replaced by the displaced area. The summation convention is implied, with the indices j, k taking the values 1,2,3 (1,2 in two dimensions). The index i is restricted similarly, and (1) cannot be used to find the hydrodynamic moment.

In many practical problems it is desirable to have a similar approximation for the hydrodynamic moment. For example, studies of the capsizing of small vessels in large waves focus on the rotational motion of the vessel, in response to hydrodynamic and inertial moments. The direct solution of such problems is extremely difficult, due to the importance of nonlinear effects, and the possibility of a long-wavelength approximation is appealing. This was the original motivation for the present work, which extends the first member of (1) in the context of the linearized exciting force and moment.

Before addressing this problem directly, we recall another fundamental relation from potential theory which states that the velocity potential ϕ_i, for forced motion of the body with unit velocity, can be approximated in the far field by three orthogonal dipoles,

$$\phi_i \simeq (m_{ij}/\rho + \forall \delta_{ij}) \frac{\partial}{\partial x_j} \frac{1}{4\pi r} \tag{2}$$

In two dimensions the source potential $1/4\pi r$ is replaced by $-1/2\pi \log r$. The mode of body motion is arbitrary, including rigid-body rotations as well as translations, but the dipole moment vanishes for rotations about the principal axes and (2) provides no information regarding the remaining quadrupoles which dominate the far-field approximation for rotational modes.

Relations similar to these are discussed extensively by Landweber and Yih (J. Fluid Mech., **1**, 319-336, 1956), with special attention to the "missing relations" corresponding to the extension of (2) to "pure rotation". For such modes it is shown by Landweber and Yih that the added moment-of-inertia for a general body shape cannot be related in such a simple manner to its effective singularities.

Hereafter F_i denotes an arbitrary scalar component of the force or moment, and ϕ_i is the "radiation" potential for forced motion in the corresponding mode of translation or rotation, with unit velocity, in the absence of an incident field. With Φ the incident-field potential, the total "diffraction" potential including the body disturbance is $\Phi + \Psi$, and the linearized pressure force or moment is given by

$$F_i = -\rho \frac{d}{dt} \iint (\Phi + \Psi)\phi_{in}\, dS = -\rho \frac{d}{dt} \iint (\Phi\phi_{in} - \phi_i \Phi_n)\, dS \qquad (3)$$

where the integrals are over the body surface and the subscript n denotes the normal derivative. (The last equality follows by applying Green's theorem to the term involving the body potential Ψ, and invoking the boundary condition $\Psi_n = -\Phi_n$.)

The next step in the analysis is to expand Φ in a Taylor series about the origin of the body. Adopting a symbolic notation,

$$F_i = -\rho \frac{d}{dt} \sum_{n=0}^{\infty} \frac{1}{n!} \iint \left(\phi_{in} - \phi_i \frac{\partial}{\partial n}\right) \mathbf{x}^n \, dS \cdot \nabla^n \Phi_{(0)} \equiv -\rho \frac{d}{dt} \sum_{n=0}^{\infty} \frac{1}{n!} \mathbf{I}^{(n)} \cdot \nabla^n \Phi_{(0)} \qquad (4)$$

Here $\mathbf{I}^{(n)}$ and $\nabla^n \Phi_{(0)}$ are multi-dimensional vectors. $\mathbf{I}^{(n)}$ is defined by the corresponding surface integrals in (4), and depends only on the body geometry and forced-motion potential ϕ_i. The time-dependent vector $\nabla^n \Phi_{(0)}$ involves n'th-order derivatives of the incident-flow potential, evaluated at the local origin of the body. Since the body is stationary, and ϕ_i is independent of time, $\nabla^n \Phi_{(0)}$ is the only time-dependent component of (4). These two vectors represent respectively generalizations of the corresponding factors $(m_{ij}/\rho + \forall\delta_{ij})$ and $\partial^2\Phi/\partial x_j \partial t$ in equation (1).

Equation 4 is the desired result, which can be applied to find the moment, as well as the force, acting on a body in a slowly-varying external flow. Before making this application, we shall generalize (2) and derive the complete multipole expansion of the forced-motion potential ϕ_i. The reason for this digression is that the duality between (1) and (2) extends more generally to the complete multipole expansion and (4).

From Green's theorem the potential ϕ_i can be expressed in the form

$$\phi_i(\mathbf{x}) = \iint \left(\phi_{in} - \phi_i \frac{\partial}{\partial n}\right) \frac{1}{4\pi R}\, dS \qquad (5)$$

The inverse of the distance R between the source point ξ and the field point $\mathbf{x}$ can be expanded in an analogous Taylor series, when the field point is sufficiently far from the body, and the result is the multipole expansion

$$\phi_i(\mathbf{x}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \iint \left(\phi_{in} - \phi_i \frac{\partial}{\partial n}\right) \mathbf{x}^n \, dS \cdot \nabla^n \frac{1}{4\pi r} \equiv \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mathbf{I}^{(n)} \cdot \nabla^n \frac{1}{4\pi r} \qquad (6)$$

where the dummy variable is changed from ξ to $\mathbf{x}$ in the last integral. Apart from the factor $(-1)^n$, the coefficient of each partial derivative (of the source potential $1/4\pi r$) in this expansion is identical to the coefficient of the same partial derivative (of the external flow potential Ψ) in the expression (4) for the force or moment. Thus the duality reflected in the form of (1) and (2) extends to the more complete expansions (4) and (6), and the force or moment due to the external field can be expressed in terms of the multipole coefficients of the corresponding forced-motion potential ϕ_i. Similar dual relations can be derived involving spherical or cylindrical harmonics, since the vector operators in (4) and (6) can be converted directly to such forms.

To illustrate this approach with a simple example, consider the rolling moment on a submerged ellipse with horizontal semi-axis a and vertical semi-axis b. For forced rolling motions with unit angular velocity, in an otherwise infinite fluid, the potential is given in elliptic coordinates by

$$\phi_6 = -\tfrac{1}{4}(a+b)^2 e^{-2\mu} \sin 2\theta \tag{7}$$

where $x + iy = \cosh(\mu + i\theta)$ (cf. Lamb, §72, eq. 7). In the far-field the leading term in an asymptotic expansion of this potential is the quadrupole

$$\phi_6 \simeq -\tfrac{1}{16}(a+b)^2 (a^2-b^2) \frac{\sin 2\theta}{r^2} = -\tfrac{\pi}{8}(a+b)^2 (a^2-b^2) \frac{\partial^2}{\partial x \partial y} \frac{-1}{2\pi} \log r \tag{8}$$

If the ellipse is submerged, with its center at a depth h below the free surface, and plane progressive waves propagate in the $+x$ direction, the external-flow potential is given by

$$\Phi = \frac{gA}{\omega} e^{-kh+ky-ikx+i\omega t} \tag{9}$$

where the real part is understood. Using this result with (4), (6) and (8), it follows that the roll exciting moment in long waves is approximated by

$$F_6 \simeq \tfrac{\pi}{8}\rho(a+b)^2 (a^2-b^2) \left(\frac{\partial^3 \Phi}{\partial x \partial y \partial t}\right)_{x=y=0} = \tfrac{\pi}{8}\rho g k^2 A (a+b)^2 (a^2-b^2) e^{-kh+i\omega t} \tag{10}$$

Note that the quadrupole coefficient in the second form of (12) appears in (14). This coefficient differs from the added moment-of-inertia of the ellipse by the factor $(a+b)/(a-b)$; hence the quadrupole coefficient and exciting moment change their signs if the major and minor axes (a, b) are interchanged, wheras the added moment is not affected.

As a three-dimensional example, consider a prolate spheroid with horizontal axis, semi-length a, and maximum radius b. If spheroidal coordinates are defined by $x = c\cosh\mu\cos\theta$, $y + iz = c\sinh\mu\sin\theta e^{i\alpha}$, the velocity potential for rotation of the spheroid about the z-axis is

$$\phi_6 = CP_2^1(\cos\theta)Q_2^1(\cosh\mu)\cos\alpha \simeq -\tfrac{2}{5}CP_2^1(\cos\theta)(\cosh\mu)^{-3}\cos\alpha \simeq \tfrac{2}{5}Cc^3 \frac{\partial^2}{\partial x \partial y}\frac{1}{r} \tag{11}$$

where P_n^m and Q_n^m are the associated Legendre functions, the constant factor C is defined by

$$C = -\tfrac{1}{3}c^2 \Big/ \frac{d}{d\mu_0} Q_2^1(\cosh\mu_0) \tag{12}$$

and $a = c\cosh\mu_0$, $b = c\sinh\mu_0$. Comparing (11) with (4) and (6), and using (9), it follows that the pitch moment on a submerged spheroid is approximated in long waves by

$$F_6 \simeq -\tfrac{8\pi}{5}\,\rho g A k^2 C c^3 e^{-kh+i\omega t} \tag{13}$$

With the additional approximation that the spheroid is slender ($b << a$), $C \simeq -\frac{1}{3}b^2$ and (13) reduces to

$$F_6 \simeq -\tfrac{8\pi}{15}\,\rho g A k^2 a^3 b^2 e^{-kh+i\omega t} \tag{14}$$

The last result can be derived from strip theory, using (2) to find the vertical force at each section and expanding (9) in a Taylor series about $x = 0$.

So far free-surface effects have been ignored, except for the incident-wave system. The same basic approach can be applied to floating bodies, or submerged bodies close to the free surface, but for (3) to be exact the potential ϕ_i must satisfy the free-surface boundary condition. (In this case (3) is essentially equivalent to the Haskind relations.) Nevertheless, some simplified asymptotic results for long wavelengths can be obtained by noting the low-frequency approximation of ϕ_i. For horizontal modes of translation and rotation about the vertical axis, the low-frequency limit of this potential is identical to that of the double body, and the synthesis used above for submerged bodies can be followed. With a multiplicative factor of one-half to account for the submerged portion of the double body, the yaw moment on a prolate spheroid which floats with its axis in the free surface is

$$F_6 \simeq \tfrac{4\pi i}{5}\,\rho g A k^2 C c^3 \cos\beta \sin\beta e^{i\omega t} \tag{15}$$

(Note that z is now the vertical axis, and β denotes the angle of the incident waves relative to the $+x$ axis.) The corresponding slender-body limit is

$$F_6 \simeq \tfrac{4\pi i}{15}\,\rho g A k^2 a^3 b^2 \cos\beta \sin\beta e^{i\omega t} \tag{16}$$

The last result also can be confirmed from strip theory, using the long-wavelength approximation for the sway exciting force on a floating circular cylinder which follows from (1).

It should be emphasized that the slender-body approximations in (14) and (16) are introduced only to verify the agreement in these limits with strip theory. The more general results (10) (13) and (15) are valid without the assumption of slenderness, and cannot be derived from (1).

The duality between the expansions (4) and (6) is illuminating, and useful in cases such as the examples above where the multipole expansion can be derived analytically. For more complicated bodies (4) can be used directly with the surface integrals evaluated numerically from the double-body potential.

Different results follow for the vertical modes of motion of a floating body, because of the opposite phase of the image above the free surface. For example, the forced-motion potential corresponding to pitching motion behaves in the low-frequency limit like a dipole, instead of a quadrupole. The corresponding contribution to the pitch exciting moment is simply hydrostatic. More complete expansions for both horizontal and vertical modes can be studied by including higher-order terms in the low-frequency approximation of the forced-motion potential ϕ_i.

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Discussion

Grue: In the presentation approximated values of the exciting moment on a submerged body are derived under the assumption that the body is deeply submerged. Often this approximation can be valid even though the body is rather close to the free surface. Can you quantify the ratio between the submergence of the body and its horizontal dimensions when this approximation is good, for instance for a very slender body?

Newman: As Dr Grue suggests, the results from analytic solutions reveal that the "deeply" submerged approximation remains valid if the body scale ℓ is comparable to the submergence depth h. In many cases a limit such as $h > \ell/2$ may be sufficient.

The appropriate choice of ℓ is less clear, for a body with disparate length scales. For a slender body it is permissible to take ℓ as the smaller transverse dimension. More generally, it may be that ℓ must be a length such that the scattering effect of the body $\to 0$ as $\ell \to 0$.

Pawlowski: I would like to remark that there is an approach to the computation of forces induced by an incident potential, which results in formulae involving added mass and damping (for bodies above to free surface) coefficients, including added moments of inertia in angular motions. In that approach (I call it "equivalent motion") the body disturbance potential is approximated by a series of radiation potentials with unknown speed amplitudes which can be determined from the impermeability conditions on the wetted surface of the body, understood in least-squares sense. In the approximation radiation potentials corresponding to other modes of distortion of the wetted surface can also be added or used, as long as they remain linearly independent, and therefore in principle an arbitrary accuracy of the flow modelling can be established.

Newman: If the normal velocity induced on the body by the incident field is expanded in a set of modal functions it is certainly possible to relate the scattering potential and exciting force to the corresponding expansions. However, it must be emphasized that higher-order modes (corresponding in the long-wavelength approximation to a pure strain of the body) are required to find the exciting moment, and the latter cannot be found in that approach from the potential due to rigid-body-rotation and corresponding added moment-of-inertia.

The essential point of my paper is that, although the rigid-body potentials are not sufficient to represent the scattering field, the quadrupole moment of these potentials is sufficient to determine the exciting moment.