SECOND ORDER DIFFRACTED WAVES AROUND AN AXISYMMETRIC BODY

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INTRODUCTION Second order wave effects have been a topic of increasing interest during the past decade, but controversies still abound even for the simplest case of the second order force on the bottom extended cylinder in regular waves. Molin (1979) derived the most consistent formulation which correctly accounts for the second order radiation condition. However in Molin's formulation, only the second order exciting force can be obtained rather than the second order diffraction potential itself which is useful for the calculation of run up, local pressure and the velocity field. Here the formulation ,which can solve for the second order diffraction potential, is developed by applying Green's theorem to the second order diffraction potential and double frequency linear Green function. The resulting integral equation is a Fredholm integral equation of the second kind which is almost identical to that of the linear problem except for the additional free surface integral. This two dimensional integral equation can be further reduced to one dimensional form for the case of axisymmetric bodies by distributing ring sources or dipoles on the body and the free surface.

STATEMENT OF THE PROBLEM Assuming potential flow and weak nonlinearity, we can use a perturbation expansion for the velocity potential. Then the boundary value problem for the second order-double frequency diffraction potential $\phi_D{}^{(2)}$ is

$$\nabla^2\phi_D^{(2)} = 0 \qquad \text{in the fluid}$$

$$(-4\omega^2 + g\frac{\partial}{\partial z})\,\phi_D^{(2)} = q \qquad \text{at } z = 0$$

$$\frac{\partial\phi_D{}^{(2)}}{\partial n} = -\frac{\partial\phi_I^{(2)}}{\partial n} \qquad \text{on the body} \tag{1}$$

$$\frac{\partial\phi_D{}^{(2)}}{\partial z} = 0 \qquad \text{at } z = h$$

radiation condition at infinity.

where the inhomogeneous term of the free surface condition consists of quadratics of linear potential and its derivatives :

$$q = -\frac{i\omega}{2g}\,\phi^{(1)}(-\omega^2\frac{\partial\phi^{(1)}}{\partial z} + g\frac{\partial^2\phi^{(1)}}{\partial z^2}) + i\omega\,(\nabla\phi^{(1)})^2\Big|_{z=0} - q_{II} \tag{2}$$

A main difference of this boundary value problem compared to a linearized one is the existence of an inhomogeneous forcing term which extends to infinity. As long as this forcing term is finite, or decays fast enough to make it absolutely square integrable (Cauchy-Poisson problem), the Sommerfeld radiation condition is acceptable. However, because of the existence of a non-decaying incident wave even at infinity, the radiation condition must be reconstructed to satisfy a free surface condition up to leading order there. Noting that the above boundary value problem is still linear, we can decompose $\phi_D^{(2)}$ into a homogeneous solution, ϕ_H ,which satisfies homogeneous free surface and inhomogeneous body boundary conditions, and a particular solution, ϕ_P , which satisfies inhomogenous free surface and homogenous body boundary conditions. The asymptotic form of ϕ_H and ϕ_P can then be written ,as suggested by Molin, as

$$\phi_H \sim \frac{e^{ik_2R}}{\sqrt{R}} + O(R^{-\frac{3}{2}}), \qquad \phi_P \sim \frac{e^{ikR(1+\cos\theta)}}{\sqrt{R}} + O(\tfrac{1}{R}) \tag{3}$$

If Green's theorem is applied to the second order diffraction potential and the double frequency linear Green function, and the following weak radiation condition is employed :

$$\iint_{S_\infty} (\phi_D^{(2)} \frac{\partial G}{\partial n} - G \frac{\partial \phi_D^{(2)}}{\partial n})\, dS \to 0 \qquad \text{as } R \to \infty \tag{4}$$

A Fredholm integral equation of the second kind can be obtained with an additional free surface integral compared to the linear problem :

$$2\pi\phi_D^{(2)} + \iint_{S_B} \phi_D^{(2)} \frac{\partial G}{\partial n}\, dS = - \iint_{S_B} G \frac{\partial \phi_I^{(2)}}{\partial n}\, dS + \frac{1}{g} \iint_{S_F} qG dS \tag{5}$$

The first term of the righthand side corresponds to the contribution of the homogeneous solution, and the second term is the contribution for the particular solution. A main difficulty of this problem lies in the succesful evaluation of the free surface integral whose convergence is very poor. Because of this highly oscillating and slowly decaying integrand, any direct truncation at some finite distance will cause significant error. In the present research, an arbitrary vertical axisymmetric body is considered, for which $\phi_D^{(2)}$, q and G are expanded by Fourier cosine series:

$$\begin{pmatrix} \phi_D^{(2)} \\ q \end{pmatrix} = \sum_{n=0}^{\infty} \begin{pmatrix} \phi_{Dn}^{(2)} \\ q_n \end{pmatrix} \cos n\theta, \quad G = \sum_{n=0}^{\infty} \frac{E_n}{2\pi} G_n \cos n(\theta - \theta_o) \tag{6}$$

Integrating eq.(5) in the circumferential direction and using orthogonality, the following one dimensional integral equation can be derived for each Fourier mode :

$$2\pi\phi_{Dn}^{(2)} + \int_{S_B} \rho dl\, \phi_{Dn}^{(2)} \frac{\partial G_n}{\partial n} = - \int_{S_B} \rho dl \frac{\partial \phi_{In}^{(2)}}{\partial n} G_n + \frac{1}{g} \int_{S_F} \rho d\rho\, q_n G_n \tag{7}$$

Here q_n and G_n are the n-th order inhomogeneous free surface term and ring source respectively :

$$q_n = \sum_{\substack{m=0 \\ (m\neq \frac{n}{2})}}^{n} Q_1 \left(\phi_{n-m}\phi_m, \frac{\partial\phi_{n-m}}{\partial z^2}\phi_m, \ldots\right) + \sum_{m=0}^{\infty} Q_2 - q_{IIn} \tag{8}$$

$$G_n = \int_0^{2\pi} G \cos n\alpha \, d\alpha \tag{9}$$

The first step of the present analysis is to develop an efficient algorithm for evaluating the ring source and its kernel to higher order. The rankine part of the ring source and its kernel can be expressed by the second kind Legendre functions of integer plus half order :

$$\left(\frac{1}{r}\right)_n = \frac{2}{\sqrt{\rho\rho_o}} \, Q_{n-\frac{1}{2}}\left(\frac{\rho^2+\rho_o^2+(z-\varsigma)^2}{2\rho\rho_o}\right) \tag{10}$$

$$\frac{\partial}{\partial n_{\vec{x}}}\left(\frac{1}{r}\right)_n = \left(n_\rho \frac{\partial}{\partial\rho} + n_z \frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right)_n \tag{11}$$

For n=0,1 , $Q_{n-1/2}$ can be evaluated from the elliptic integrals, however, due to instability of the forward recurrence, each order must be calculated seperately. In calculating $Q_{n-1/2}$ for the case n>2, two different forms of hypergeometric series are found ,which are appropriate in each specific region, and they are converted into Chebyshev economizing polynomials for efficient calculation.
An inverse discrete Fourier transform is used for the evaluation of the non -singular ring source with finite number of spectral points M, and the related error can be measured from the (M+1)-th term which converges to zero rapidly with incresing M. To calculate the linear potential and its first and second derivatives on the free surface more directly, the ring source distribution method is used in preference to the combined distribution. After evaluating q_n and G_n, the next step ,which is the most critical and time consuming, is the calculation of the infinite domain free-surface integral whose oscillating amplitude decays as the inverse of the square root of the truncation point. For this calculation we subtracted out the far field asymptotic of the integrand and integrated it analytically :

$$I_{FS} = \int_a^b \rho d\rho q_n G_n + \int_b^\infty \rho d\rho (q_n G_n - \hat{q}_n \hat{G}_n) + \int_b^\infty \rho d\rho \, \hat{q}_n \hat{G}_n \tag{12}$$

We used following asymptotics for the ring source and ring potential, which are exact outside the region where all local modes vanish :

$$\hat{G}_n \sim C_o \cosh k(z+h) \cosh k(\varsigma+h) J_n(k\rho_o) H_n(k\rho)$$
$$\hat{\phi}_n \sim C_o \cosh k(z+h) H_n(k\rho) \int \rho_o dl_o \, \sigma_n J_n(k\rho_o) \cosh k(\varsigma+h) \tag{13}$$

We can then derive the far field asymptotic of the integrand whose typical terms are triple products of Bessel and Hankel functions. If we use Chebyshev economizing polynomials of the Hankel function, which is valid outside the truncation point, a contribution of the second integral in eq.(12) can be neglected ,and the third integral can be expressed by triple summations which include various kinds of related Fresnel integrals.

As a result we can shrink the upper bound of the numerical integration to the limit of the local modes-free region. An alternative and much easier approach is to use the leading asymptotic of the Hankel function. However numerical experience shows that as the order increases, we have to go further out on the free surface for a reasonable truncation, which results in huge computing time.

NUMERICAL RESULTS

In solving the integral equation, the body contour is divided into straight line segments and the strength of the ring source is assumed to be constant over each panel. Numerical convergence of the linear diffraction problem is tested for the bottom extended vertical cylinder by increasing the number of segments and spectral points ,and three or four decimal accuracy is achieved. A semi-analytic expression for the contribution of the second order diffraction potential to the force on the bottom extended vertical cylinder, can be obtained by using the explicitly known linear diffraction potential and the double freqency radiation potential. The horizontal force calculated from the first mode of a second order diffraction potential by the present method is compared with this semi-analytic solution and there is good agreement, as shown in figure 1. Second order wave run up around the cylinder ,including the contribution of a second order diffraction potential, is compared with the linear run up in figure 2.

REFERENCES

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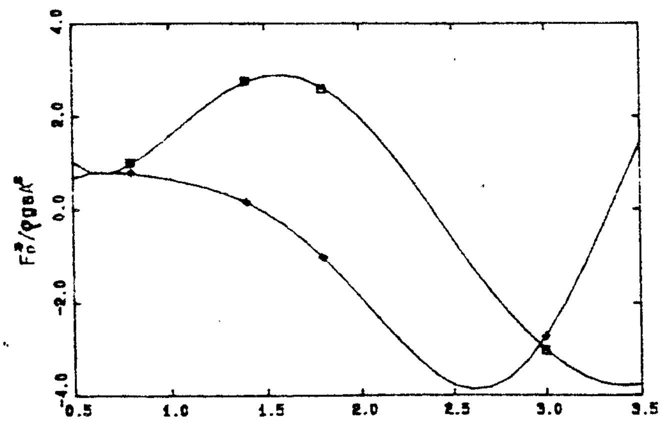

Fig 1 : Horizontal force on the bottom extended cylinder due to the second order diffraction potential. (——): semi-analytic solution, real(■) & imag(◆): present method

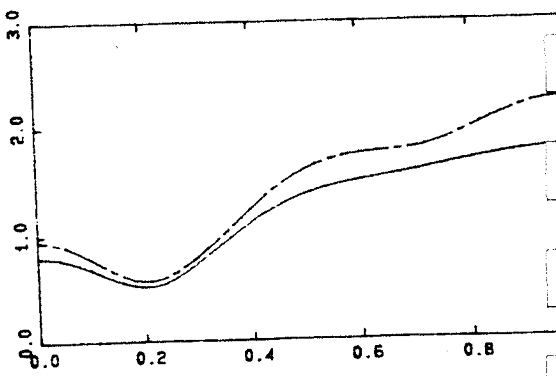

Fig 2 : Linear(——) and second order(-·-) wave run up around the cylinder.

Discussion

Papanikolaou: It is not clear to me why the integrand of the free-surface integral doesn't converge rapidly. In fact the disturbances in the 3D case, corresponding to waves in the horizontal plane, should decay much quicker than in the equivalent 2D case, where the disturbance is confined in one direction only. It has been no problem in the past to solve the second-order 2D problem in a satisfactory manner (e.g. Papanikolaou et al, ONR, 1980). You might have some irregular frequencies problems in the studied frequency regions but these frequencies can be estimated by formulas in the case of simplified bodies, like the vertical cylinder.

Kim: The poor convergence of the free surface integral is due to its highly oscillatory nature. A simple moving average technique can be used unless high accuracy is required. The 2nd order problem is more sensitive to the irregular frequency because even the right hand side is affected by that of the 1st order problem.

Eatock Taylor: A student of mine, F.P. Chau, is also working on this problem, and has obtained results for a vertical cylinder. The principles of the formulation are similar to Kim's, although details of the free surface integral differ. For the cylinder Chau's results are based on a semianalytical approach rather than on a ring source distribution. He has compared the run-up predicted by Kim with his own results, and finds similar but not identical results. Bearing in mind the rapid variation of second-order pressure with depth, one feels that the arrangement of constant source segments must be very important near the free surface. Could the author comment on the convergence of his mesh?

Kim: The size of the panel or mesh should be very fine for the second order problem, especially near the free surface. It is worthwhile to compare the results calculated from different methods. The rapid variation of the pressure near the free surface is accounted for in my program by cosine spacing.