NONLINEAR IMPULSIVE PROBLEMS

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Introduction

Hydrodynamic loadings acting upon ocean structures have classically been computed using linear theory. Even if second order approximations exist or are being developed, the limitations of linear or weakly nonlinear models are now recognized in many situations, and especially for the prediction of the behavior of floating ocean structures in extreme waves.

With the availability of supercomputers, approximate methods have been developed to solve numerically the exact nonlinear equations in the time domain [1]. Even though these methods are still time-consuming and therefore limited to two-dimensional or axisymmetric flows, remarkable results have been obtained, for instance for the prediction of overturning.

However, difficulties remain. In particular when free surface piercing bodies are present, the behavior of the flow in the vicinity of the waterline is still far from being understood. We believe that this problem should be analyzed mathematically prior to develop any reliable numerical algorithm to predict the nonlinear motion of a floating body.

A simple configuration to study this problem is the motion of a vertical wavemaker in a tank of finite depth H, a problem already studied analytically, numerically and experimentally by Lin, 1984 (see references herein contained), and which is now becoming a standard for two-dimensional numerical wave tanks. After a general discussion of the problem, we will consider the impulsive wavemaker problem and study the difficulties associated with it.

The Wavemaker Problem

We consider a semi-infinite tank of depth H, with a vertical wavemaker at its end. The wavemaker is moving with a velocity scale U and a frequency scale ω. An asymptotic study of the significant degeneracies of the Free Surface Boundary Condition (FSBC) in the vicinity of the waterline has been performed, leading to the results shown below, valid when the depth of the tank is sufficiently large (i.e. larger than the length scale indicated).

ACCELERATION OF THE WAVEMAKER	DOMINANT TERMS IN THE FSBC	LENGTH SCALE
$U\omega/g \ll 1$	linear and gravity	g/ω^2
$U\omega/g = O(1)$	all	g/ω^2 or U/ω
$U\omega/g \gg 1$	all except gravity	U/ω

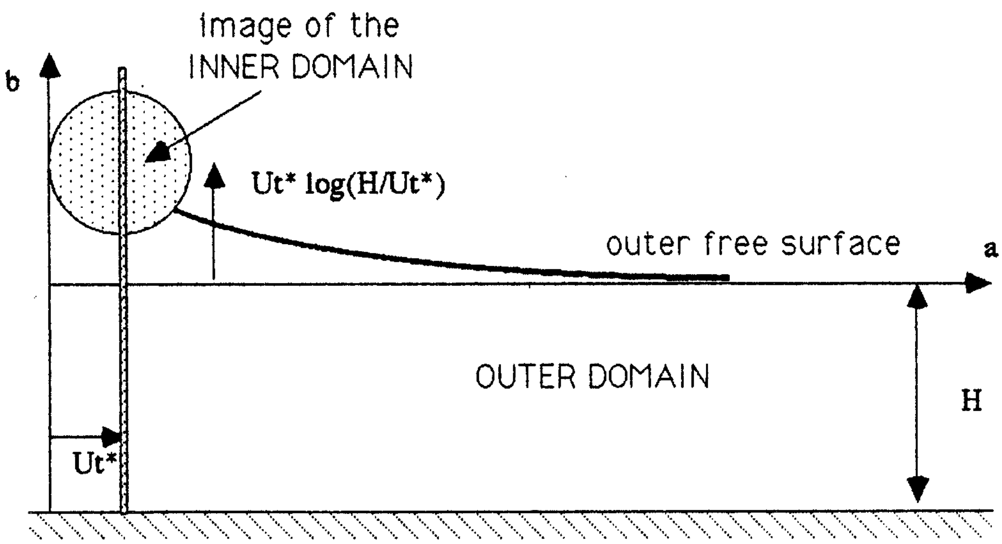
These results let expect that the linearization of the free surface boundary condition in the vicinity of the wavemaker is legitimate if - and only if - $U\omega/g$ is (much) smaller than 1. They seem to be well supported by analytical, numerical and experimental observations.

[1] But still within the framework of potential flow theory. Real fluid effects will not be considered here.

A mathematical study of impulsive flows - i.e. of flows for which the acceleration of the wavemaker is much greater than the acceleration of gravity - appears necessary. It should however be noticed that these problems, even at first order, are expected to be fully nonlinear and transient. They are therefore unlikely to be solved analytically, even in an asymptotic sense. A further simplification is needed, and will be sought using self-similarity. This implies that the fully impulsive problem will be considered (where the acceleration of the wavemaker is a Dirac delta function). The validity of such an idealization will be discussed later.

The Impulsive Wavemaker Problem

We now assume that the wavemaker is fixed for $t^* < 0$ [2] and moving at constant velocity U for $t^* > 0$. For a depth much greater than the displacement of the wavemaker, the length scale in the vicinity of the wavemaker is expected to be Ut^*. We therefore expect a self-similar solution to exist, as for the impact at constant velocity of a wedge on an initially flat free surface. However, self-similarity is only possible here in an inner domain and an outer solution taking into account the bottom boundary condition has first to be found [3].



INNER AND OUTER DOMAIN FOR THE IMPULSIVELY PUSHED WAVEMAKER (SCHEMATIC)

The Outer Problem

We define $\delta = H/D$, where H is the depth of the tank and D scales the horizontal displacement of the wavemaker. The actual displacement of the wavemaker is εUt^*, where $\varepsilon = +1$ if the wavemaker is pushed and $\varepsilon = -1$ if the wavemaker is pulled. Taking H as length scale, U as velocity scale and D/U as time scale, the non-dimensional equations for the full nonlinear problem (including gravity) are readily written using a Lagrangian specification. The unknowns are the non-dimensional displacements (X,Y) of a particle originally located at (a,b). The outer solution is found in the limit where $\delta \to \infty$ [4]. It is identical to the solution derived by Lin (1984) and has the classical logarithmic singularity near the origin where ($i^2 = -1$):

[2] t^* is the dimensional time, t is the non-dimensional time, Ut^*/D.

[3] The inner domain is defined in the vicinity of the waterline, i.e. in the vicinity of $(a,b) = (0,0)$. The outer domain is defined on a length scale H.

[4] Gravity can be neglected at the leading order in the outer domain provided that $gH/U^2 << \delta^2$.

$$X - iY \approx \frac{\varepsilon}{\delta}\, t\, \frac{2}{\pi}\, i \log\left(\frac{\pi}{4}[a+ib]\right) \text{ as } |a+ib| \to 0 .$$

The classical linearized solution for small time appears therefore here as an outer solution demanding to be matched to an inner solution defined in the vicinity of the waterline.

The Inner Problem

Since a significant degeneracy is expected in a domain of length scale D near the origin, and since we are looking for a self-similar solution, we define inner variables by :

$$\bar{a} = \frac{\delta a}{t}, \quad \bar{b} = \frac{\delta b}{t}, \quad \bar{X} = \frac{\delta X}{t}, \quad \bar{Y} = \frac{\delta Y}{t} - \varepsilon \frac{2}{\pi} \log\left(\frac{4\delta}{\pi t}\right),$$

where the last term on the RHS corresponds to the logarithmic singularity of the outer solution. This physically defines the image of the inner domain as a domain of length scale Ut^*, in contact with the wavemaker and located at a height $\varepsilon\, Ut^* \log(H/Ut^*)$ above the initial water level (see figure).

The equations resulting from this change of variable and the matching condition are easily obtained at the leading order [5]. They correspond to the equations for the water entry problem in Lagrangian coordinates (Johnstone & Mackie, 1973), except for :

- the wavemaker boundary condition,

  $\bar{X} = \varepsilon$;

- the Free Surface Boundary Condition,

  $$\bar{a}^{-2} \bar{X}_{\bar{a}\bar{a}} (\bar{X}_{\bar{a}} + 1) + \bar{a}^{-2} \bar{Y}_{\bar{a}\bar{a}} \bar{Y}_{\bar{a}} - \varepsilon \frac{2}{\pi} \bar{Y}_{\bar{a}} = 0 ;$$

- the behavior at infinity,

  $$\bar{X} - i\,\bar{Y} \to \varepsilon \frac{2i}{\pi} \log(\bar{a} + i\,\bar{b}) \text{ as } \bar{a} + i\,\bar{b} \to \infty .$$

Using asymptotic expansions and self-similarity, it has therefore been possible to reduce the equations governing the evolution of the flow in the vicinity of the waterline to a steady problem in an appropriate coordinate system. Note in particular that the behavior at infinity and the wavemaker boundary condition are consistent.

Unlike the water entry problem, self-similarity does not imply the conservation of the arc length along the free surface – see the FSBC. In fact, it seems that the term related to the vertical acceleration of the inner domain imposed by the outer solution (the last term of the LHS of the FSBC) has a major importance. This can be further analyzed by considering the linearized inner problem.

The Linearized Inner Problem

Assuming that the gradients of the displacements are much smaller than 1, the inner problem can readily be solved by use of the Fourier transform. This yields :

$$\bar{X} = \varepsilon - \frac{1}{\pi} \int_0^\infty A(\alpha) \exp(\alpha \bar{b}) \sin(\alpha \bar{a})\, d\alpha , \quad \text{with } (\alpha^2 A)_{\alpha\alpha} - \frac{2}{\pi} \varepsilon A \alpha = 0 \ (\alpha > 0) ,$$

X and Y being harmonic and satisfying the Cauchy conditions. Solutions of the last equations are given in terms of Bessel functions (J_1 and Y_1) for $\varepsilon = -1$ and modified Bessel functions (I_1 and K_1) for $\varepsilon = +1$.

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[5] Gravity can be neglected at the leading order in the inner domain provided that $gH/U^2 \ll \delta$.

Matching in the Fourier domain yields a unique solution in each case. If this method leads to a satisfactory solution when the wavemaker is pulled, a major difficulty arises when the wavemaker is pushed. In this case, the solution in the Fourier domain increases exponentially at infinity and an inverse Fourier transform cannot therefore be performed. This lets expect (at least) a strong singularity at the origin in the physical domain and has still to be explained. It might be due to the linearization of the inner problem, but the possibility of an ill-posed self-similar problem cannot be excluded, as this is discussed in appendix for a simple example.

Conclusion

As for the water entry problem, self-similarity does provide a powerful tool to deal with nonlinear impulsive problems. For the impulsive wavemaker, self-similarity is however only possible in an inner domain and matching with an appropriate outer solution taking into account the bottom boundary condition is required.

The procedure used is consistent and seems to yield a uniformly valid solution when the wavemaker is impulsively pulled. However, no regular solution has been found when the wavemaker is pushed. A possible reason for this difficulty is that the impulsive wavemaker problem might be ill-posed, i.e. that the full transient nonlinear problem (possibly without gravity) might have to be solved because, even when the acceleration of the wavemaker approaches a Dirac delta function, it cannot be reduced using self-similarity. This possibility is considered in appendix analyzing a simple example.

Appendix : An Impulsive Problem without Regular Solution

We consider a nonlinear single degree of freedom system excited by a delta Dirac function :

$$Y_{tt}\, Y = \frac{d^2}{dt^2}\{\, t\, H(t)\,\} \quad \text{with} \quad H(t) = 0 \ \text{for} \ t < 0 \ , \quad H(t) = 1 \ \text{for} \ t \geq 0 \ .$$

The question arises to know whether or not this problem is well-posed. We therefore consider a series of functions f_γ such that

$$f_\gamma(t) \rightarrow t\, H(t) \ \text{as} \ \gamma \rightarrow 0 \ ,$$

i.e. we formulate the transient problem and we let the rise time go to zero. If we choose for f_γ :

$$f_\gamma(t) = 0 \ \text{for} \ t < 0 \ , \quad f_\gamma(t) = \frac{t^4}{4\gamma^3} \ \text{for} \ 0 \leq t \leq \gamma \ , \quad f_\gamma(t) = t - \frac{3}{4}\gamma \ \text{for} \ \gamma < t \ ,$$

a solution of the transient problem appears to be :

$$Y_\gamma(t) = \sqrt{\frac{6}{\gamma}}\, t + \sqrt{\frac{3\gamma}{2}} - \sqrt{6\gamma} \ , \quad \gamma < t \ .$$

Surprisingly enough, as $\gamma \rightarrow 0$, Y_γ has no finite limit. The solution is a function of the transient phase and is singular when the duration of the transient phase goes to zero. A similar phenomenon might occur for the "impulsively" pushed wavemaker.

References

Johnstone, E., & Mackie, A., 1973, "The Use of Lagrangian Coordinates in the Water Entry and Related Problems," Proc. Camb. Phil. Soc., Vol. 74, p. 529.

Lin, W.-M., 1984, Nonlinear Motion of the Free Surface near a Moving Body, Ph.D. Thesis, MIT, Cambridge, Mass. (also see references herein contained).